ECONOMIC ANALYSIS OF PROVISION OF RELIABILITY IN WEAPONS

by

John N. Green

Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Arts
1968

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ABSTRACT

Title of Thesis: Economic Analysis of Provision of Reliability in Weapons

John N. Green, Master of Arts, 1968

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The problem analyzed is how to provide most efficiently in a weapon, in an economic sense, high reliability - defined as the probability of the weapon to operate properly whenever, within its prescribed service life, it is called upon to do so. The problem is discussed in the context of a re-entry body (weapon) for a strategic missile. A method is proposed for building up a supply curve of initial (when new) reliability, using a generalized Lagrange multiplier method. Selection of the reliability goal based upon the value of the product is discussed. A model is also developed for maintenance of reliability in the presence of degradation with time. The nature of this latter question is the acceptance of degradation of weapon output versus the cost of maintenance activities. Important parameters are examined in a cost minimization formulation using a standard Lagrangean format. The analysis is extended into subjective consideration of utility of reliability, and risk and uncertainty. How the analysis could be used to assist a weapon development agency in design and maintenance decisions is indicated.

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I. BACKGROUND

A. Nature of the Problem

The problem to be investigated is how to provide most efficiently, in an economic sense, a high probability of a weapon containing a complex functional device to operate properly whenever, within its prescribed service life, it is called upon to do so. We are speaking of explosive weapons which are only called upon to operate once. The time at which a weapon is used may be when almost new or after it has undergone a long period of storage. "Long" in this case means long enough that a predicted low failure rate under a relatively benign storage condition becomes significant in reduction of reliability at the time of use.[1]

The intent of this thesis is to explore how the standard tools of economic analysis may be applied to assist in a decision process concerning provision of reliability which cannot properly be the domain of reliability and systems engineering specialists alone. Some of the important questions to be decided in arriving at a weapon system of given effectiveness at least cost are:

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[1]Thus, already we see that the reliability definition above is open to ambiguous interpretation, dependent on what has been the past history of the weapon at the time of use. Contractors usually like to be held to only meeting what is called an initial (when new) or inherent reliability. Inherent has the broader connotation of meaning what the design would be capable of producing if everything in the production process were done perfectly. But then this reliability is stated to pertain initially, and the prediction of degradation before use (while it is in the hands of the using service, albeit under known specified procedures and environmental conditions) is a complex question the answer to which the contractor would not like his fee to depend upon.

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(1) Should the design and support concept be that of a "wooden bomb"? By wooden bomb is meant a weapon for which neither test not maintenance is necessary after its introduction into the operational inventory of a military service. Under this concept, to meet a reliability goal applicable after a long service life and given that reliability does degrade with time, one must pay the extra cost of development of a system with higher reliability when equipment is new. But there are obvious counterbalancing savings from the absence of test and maintenance expenses.

(2) If periodic tests are to be conducted, what are the nature and frequency of the tests? The trade-off here is between the cost of obtaining information and the value of the information obtained. Decision rules are needed concerning action upon completion of test - continue in use, repair, or replace.

(3) If the intended support concept is to treat the weapon like a wooden bomb, should allowance be made in the design for a change in the planned logistics sequence should the expectation of conditions upon which the decision was based be proved wrong in such a way as to impose a severe penalty for not having a test and maintenance activity? The question here is analogous to whether or not to pay an insurance premium for possible failures during the service life - failures whose probability may be known much less exactly than events based on insurance actuarial tables.

(4) What should be the reliability goal of the weapon, considering the possibility of a repair/replace cycle or not? Although the problem in reliability engineering, particularly from the standpoint of the reliability when new, has been extensively treated by sophisticated

optimization techniques trading off attainment of high reliability versus performance degradation and the costs of producing high reliability equipment with attendant comprehensive test programs; the solutions, especially in weapons for expensive strategic missile systems, have been rather trivial and arbitrary from an economic point of view. The worth of the weapon and supporting missile system has been so high that the reliability goal for the functional device in the weapon has been set very high, near the level of estimated feasibility with little regard for cost. But costs are expended in an attempt to meet the reliability specification, and in that way of doing business there is not likely to be an attempt made to see that marginal cost is equated to marginal revenue. A trend toward multiple weapons on each missile should enhance the importance of the economic costs of attaining reliability relative to weapon worth since the total value of the system, although it may be somewhat larger, is spread over many more weapons, reducing the value per weapon.

(5) Should the available design alternatives for the functional device, based on acceptably proved technology, be deemed unsatisfactory, suggesting that a different type of device be developed for the weapon? Technological considerations may have dictated use of an electronic device of considerable series complexity. Rather than expend effort in raising the reliability of this design, a better alternate solution might be to invest in a research and development program for a device utilizing a different technical approach with a potential for higher reliability at the same or lower cost to do the same job.

B. Failure Rate Prediction

Reliability of a device is usually conceptually considered in two different ways, dependent on whether the device must operate over a long period of time or is a one-time use piece of equipment. For an explosive mechanism usually the latter applies and a prediction of reliability would be based on the result of many trials of hopefully identical mechanisms, with reliability estimated by the number of successes divided by the total number of trials. For prediction of reliability for devices which must operate for a finite length of time in order to meet the definition of mission success, the reliability may be predicted prior to experimental testing of the device if one has a basis for predicting the probability distribution of failures over time. An estimation of failure rate expressed mathematically leads to a calculation of reliability based upon the estimate. Prediction of the failure rate of a complex functional device is based upon what is known or assumed about the failure rates of the parts making up the device and how these parts are assembled together. Electronic devices will be used for illustration in this thesis both because of their practical importance and their amenability to mathematical modeling and prediction of failure rates. However, it should be remembered that the economic theory does not depend on the physical nature of the device. It will also be evident that the decisions of major economic importance are not dependent on very accurate predictions of failure rate.

This thesis does not dwell on the engineering aspects of the problem although they are of prime importance in any practical application. Reliability prediction procedures for electronic equipment

have been extensively investigated. Reference [1] [1] is cited for a comprehensive treatment of electronic design and reliability engineering; reference [2] for reliability engineering as related to maintainability and system effectiveness. Data on failure rates of electronic parts for predictions of equipment reliability on military contracts are compiled in reference [3]. A consistent set of terms and symbols will be used in this thesis, not necessarily consistent with that used in any of the references cited.

The reliability of an electronic device is inversely proportional to series complexity, but considerable complexity may be necessary to obtain desired performance characteristics. Given the design concept, a prediction can then be made of a reliability which might be attained when the equipment is new. Usually in the past a prediction of failure rate during dormant storage has been considered too difficult to be done or deemed not significant. However, a recent report of a study based on extensive data gathering sheds more light on storage effects on electronic equipment and part reliability, indicating the feasibility of meaningful quantitative prediction [4]. If this proves to be valid and the dormant failure rates are significant, it is important that weapon system planning take them into account.

Assuming a Poisson distribution of failures over time, the reliability of a series system of electronic parts is expressed by an exponential model:

$$R = e^{-\lambda t}$$

---

[1] Numbers in [ ] will refer to the list of literature cited at the end of the thesis.

where

R = reliability of the system

e = base of natural logarithm

λ = constant failure or hazard rate under a specified set of conditions

t = time under the specified conditions.

For a weapon we will consider a factory-to-target sequence (FTS) with time phases which have different associated failure rates, dependent on the associated environmental or use conditions, and whether the equipment is operating or non-operating. Thus, for each time period of index i, we have

$$\lambda_i = k_i u_i$$

where

u_i = basic failure rate under environmental conditions defined as standard

k_i = the failure rate adjustment or stress factor for a given use condition

Then,

$$R = e^{-\sum \lambda_i t_i}$$

consistent with a failure rate addition rule of independent events. In this paper we will call the product for any period the hazard product, defined as

$$h_i = \lambda_i t_i \ .$$

Predicting the failure rate of a new configuration has been called "a vague and imprecise process at best" [5]. But the author of the remark shows the value in estimating the uncertainty involved, and it follows that a meaningful estimate is better than no estimate at all.

C. Measure of Effectiveness

To do an economic analysis of a weapon system we must decide on a measure of effectiveness (MOE), the output of the weapon's production function. System effectiveness of a weapon system is generally considered to be a product of the following three items:

(1) Reliability

(2) Performance characteristics, given no failure

(3) Availability or proportion of time ready for use.

To consider whether we are using the proper MOE in our analysis we have to talk about suboptimization and proximate criteria. These are discussed in Chapter 2 of reference [6] in relation to the manufacturing industry. The same principles apply to our production of reliability in a weapon. It is brought out that a manufacturing department for practical reasons may have to optimize at its own level although ideally the problems should be solved according to the purpose of the over-all organization. The criterion a department uses which is not explicitly linked to over-all objectives is labeled a proximate criterion. The proximate criterion should be consistent with a good criterion at a higher level. It is clear then that reliability of a weapon is only a proximate criterion. In fact, effectiveness of the weapon based on the three items above is not an overall criterion if the weapon is only a part of the weapon system. We must be careful when using a proximate criterion such as reliability of a weapon to take account of the interaction of our activities to increase reliability on the other elements of the MOE.

In this thesis which will primarily for illustrative purposes consider the reliability of weapons in the context of a strategic missile system, the use of the effectiveness of the weapon deployed

by the missile as a proximate criterion for the effectiveness of the entire missile system seems reasonable in making decisions concerning design of the carried weapon. The weapon containing a complex functional device for causing its detonation at the proper time in the trajectory will be called the re-entry body (REB). Since the rest of the missile system will in general involve more complex functions than the REB (for example, propulsion and guidance), we can in looking at sub-optimization of the effectiveness of the REB disregard explicit use of the availability term. That is, availability of the REB will be maintained at 100% of missile carrying capability, utilizing scheduled down time of the missile to accomplish any planned maintenance of the REB. (This does not say we should not worry about non-availability of REB's which might result from unexpected deficiencies in design which are subsequently brought to light by some testing program.) The extra REB's necessary to allow a planned level of maintenance which might reduce their availability in the ordinary sense of the word would appear in increased costs of procurement - not in reduction of effectiveness.

The above treatment of availability might seem to be untrue for tactical weapons as opposed to strategic ones, but in some cases it would be quite similar. Although in theory resupply of weapons can take place, a given battle is usually fought with what is on hand. For example, if we consider a shipboard surface-to-air missile (SAM) for fleet air defense, only a given number can be available on board for any engagement. Effectiveness during any battle is only reduced by the availability term if the maintenance concept calls for testing the functional devices on board for a decision as

to their condition, with or without provision for repair on board if condition is deemed bad. If the concept calls for issuance to the fleet of a "wooden round" (completely assembled SAM with no further tests called for), availability on board will be 100% with effectiveness reduced only if this concept results in lowered reliability.

The REB influences on performance to be considered are those which affect explosive yield of the contained warhead and accuracy on target. Anything which increases weight and volume of the functional device detracts from yield, due to robbing space and weight allocation for the warhead. It also detracts from accuracy because it restricts design options for the REB.

D. Analogy with Economic Theory

Since the stated intent is to apply standard tools of economic analysis to the problem, the analogy between a supply of reliability and a more usual economic good will be pointed out here.

In building up a supply curve of reliability from cost curves we can relate to the classic article by Viner [7] which treated the supply curve of firms as dependent upon different types of technological and pecuniary cost situations under the assumption of atomistic competition. His stated purpose of presenting formal types of relationships which could be conceived to exist under certain simplifying assumptions to help answer the question - Which cost curve is the proper supply curve? - is a valid objective here. Where Viner speaks of a firm within an industry producing a product, we will consider the processes or activities within the entire set of productive services which combine to supply units of weapon reliability

(looked upon as an increase of weapon reliability, ΔR, above some basic level because reliability must refer to some specific design). Where Viner speaks of price of product, we will use the value of unit ΔR. The items of interest will be the increasing portions of the average and marginal cost curves and how they relate to price of product. The practical nature of the pecuniary and technological cost situations of improving reliability puts one immediately into an area of increasing average costs, which is probably implicitly recognized by the development agencies. But do they consider the much higher marginal cost which should be examined for intersection with the price level to determine the quantity of ΔR produced?

The average and marginal costs are of course derived from the total cost function. The factors which lie behind the total cost function in the theory of the firm in relation to production decisions and the utilization of resources are discussed in Chapter 7 of reference [8]. Starting with: (1) a production function which relates outputs to inputs, (2) prices of product and inputs, and (3) the assumption that profit is to be maximized subject to the constraint of the production function; nine theorems are derived which correspond to the rules of standard marginal economic analysis. In our case the single output is ΔR of the weapon. The inputs are the various activities for which the cost of each will be translated into monetary terms. In this format the only resource to be allocated is money. For optimal decisions one of the theorems proved is that the marginal product per dollar cost is equal for every input. That is, the marginal ΔR per dollar cost of each reliability improvement process should be equal. Another theorem proved states that any

output will be produced such that its selling price equals marginal cost. This corresponds to the price (value) of unit ΔR determining the level of ΔR so that marginal cost of ΔR does equal price. Similar analogy can be made with the other theorems.

Two concepts from price theory, both discussed in Chapter 7 of reference [9] should be introduced - opportunity costs and external diseconomies. Opportunity cost for producing a given amount of product is the amount of another product which could have been produced with the given marginal amount of input. We will use the term to indicate the amount of another performance aspect which would have been obtained if we had not spent a given amount on ΔR. Viewed another way, consideration of the costs involving the overall MOE versus the costs when considering reliability alone corresponds to a social cost versus an individual industry cost. We can put these costs in monetary terms if we can place a value on the various performance aspects. Secondly, external diseconomies may result from expansion of output overall. In production theory a monetary diseconomy pertains if the price of inputs rise or a technological diseconomy pertains if average costs rise for other reasons. In our case the primary reason for an individual reliability effort to be more expensive as the aggregate effort rises is the limited amount of ΔR possible. Even if not directly conflicting in their attainment of objectives, the impact of the activities accentuates the importance of diminishing returns as reliability to be gained approaches zero.[1]

[1]In the reliability improvement problem direct analogy with opportunity costs and external diseconomies of standard production theory is somewhat strained. The two concepts might more properly be lumped together and called "external effects"; that is, activities to improve reliability have effects on other items of the ultimate MOE in addition to their effects on each other. It is even conceivable that these effects could be so large in a negative direction that the relation between reliability and weapon system effectiveness would not be a monotonically increasing function. In this thesis the costs from these external effects will be called opportunity costs, but it should be noted that there is not an exact parallel to the opportunity costs of standard economic analysis.

The processes or activities of the development agency to improve weapon reliability, which relate to the productive services of a firm, will be categorized as follows:

(1) Functional Design

- (a) Reduction of complexity
- (b) Redundancy
- (c) Improvement of insensitivity to parameter changes
- (d) Provisions for test in field

(2) Packaging

- (a) Provisions for integrated test in manufacture
- (b) Accessibility for test and repair/replace in field
- (c) Environmental stress protection

(3) Manufacturing and Quality Control

- (a) Use of high reliability parts
- (b) Process control and testing of weapon

(4) Operational Procedures

- (a) Test and repair/replace cycle
- (b) Mitigation of stress factors

These activities are chosen to relate primarily to the expected reliability of the weapon rather than to the confidence with which that reliability is predicted although the two aspects are intimately related. For instance, the activities under item (3) above involve quantities of tests, many of which will be determined by statistical confidence limits based on some assumption of probability distribution of attributes. But the primary purpose of the activity is to increase expected reliability in the ultimate use. We have not included operational tests (OT) as a category - the OT being a near as full-scale test as possible under as near actual use conditions as possible - the real purpose not being to improve reliability of the weapon but to increase confidence that the assumptions which led to the predicted reliability were correct. That is, the OT does not just increase statistical confidence in the ordinary sense although every sample contributes.

but it gives qualitative confidence that the lower level test programs were testing piece-meal the proper attributes to insure over-all success in an operational mission. (The Operational Procedures test category, 4a above, refers to a test or monitoring operation on the functional device in the field with the explicit intent of discovering and correcting bad devices before use.) Nor have we included a stock-pile sampling program (SSP) which would make thorough, intensive investigations (usually destructive tests) of small samples from the operational inventory. The SSP also makes a small contribution to the statistical confidence of reliability, but it primarily would forewarn of an impending degradation of reliability. As such an SSP has no effect on reliability unless further action is taken based upon the information provided.

Now, having indicated that we will consider the activities primarily in terms of expected reliabilities, we will find it necessary to extend the analysis into a more subjective area in order to make final common sense decisions. The two major categories, both widely used in economic theory, are:

(1) Risk, Uncertainty and Profit

(2) The Utility Analysis of Choices Involving Risk.

The first category is the title of the 1921 book by Knight analyzing profit theory from largely a macroeconomic point of view, but Chapter 7 of which is pertinent here in its discussion of the meaning of risk and uncertainty [10]. The second is the title of a 1948 article by Friedman and Savage which describes utility functions of individuals for the purpose of explaining behavior under the postulate of maximization of expected utility [11]. The pertinent portion therein is

the shape of the utility function of one who will pay a premium to avoid a large loss. We will turn the argument around; rather than explaining behavior, we will indicate what the rational behavior of the development agency should be (design decision alternatives) based upon an assumption of at least the gross shape of the ultimate user's utility function for the product. The above two references and many others are cited in the 1965 study by Fellner of economic behavior along Bayesian lines [12]. We will use only a few of the ideas from this large field of decision making under uncertainty in an attempt to have our profit maximization make sense in accordance with utility theory and in the presence of subjective probability.

Next let us define terms for use in this thesis (which as in reliability theory cannot be consistent with all the references) and indicate the relation to the particular subject. Risk applies where the probability of an event can be calculated in terms of a standard process (heads or tails type) even though the values assigned to the standard process may be somewhat subjective. Basically, risk is calculated by frequentist methods - observing an event many times and relating probability to observed frequency. It seems logical then to put failure rate prediction for electronic devices into the risk category. Even though the particular device has not yet even been built, the reliability calculations are based on piece part experience with failure rates derived by frequency methods. Uncertainty, on the other hand, applies where a method of calculation of probability of an event is less clear - sometimes said to be a case of no theory for calculation. But the subjectivist approach allows for appraisal of probability values and using them to inject judgment of this nature

in the decision process. We will use uncertainty to apply to the appraisal of failure modes which would result in reliability values beyond the scope of which the mathematical models will predict. For example, developers of complex functional devices for weapons where the specification for reliability is well above 90% will never show a mathematical model for a meaningful prediction of the probability of achieving only 20%. But we know that such things, verging on catastrophic in terms of mission success for the weapon system, have happened; and people with experience can be expected to make meaningful appraisals of the likelihood of such happenings. If one recognizes the appraisal of the probability of unpredicted failure modes, the result may be a decision to select alternatives which entail real and opportunity costs (buying insurance) to make it less costly to correct unsuspected defects should they occur.

Essentially then we can list three levels of refinement in maximizing the profit from reliability provision activities:

(1) Maximize value of expected reliability minus the cost of providing it where we assume constant value of units of reliability.

(2) Maximize the expected utility (ex ante profit) of the reliability provision where ex ante profit is theoretically defined as follows:

Let

G = marginal utility of potential gain

L = marginal disutility of potential loss

X = units of reliability provided

Then

$$\text{ex ante profit} = \int (G - L)\, dX$$

where the potential gains and losses are calculated using the risk probabilities. This calculation will not be done because the utility function is not known but qualitative judgments can be made from postulation of its general shape.

(3) Extend the above analysis to allow use of subjective probabilities.

II. PROVISION OF INITIAL RELIABILITY

A. Subsystem Reliability Allocation

To begin taking one step at a time in attacking a very complex problem let us first consider how we might arrive at a reliability goal for the REB, one subsystem of the missile, abstracting from any consideration of degradation with age. That is, we consider failure rate as zero during any period of the FTS except time in missile flight. This facet of the problem will be considered in Chapter III.

For a weapon system made up of several subsystems the general problem is that of allocating reliabilities to the subsystems so that overall reliability is attained at minimum cost. In terms of production theory the output, system reliability, is a function of the inputs, the reliabilities of the various subsystems, with each of the latter having a given cost function. The problem is very non-linear and complex. The design options are not only non-linear with cost but are usually non-continuous functions. The various alternatives involve quantum jumps. There are also interdependencies among the subsystems in the production function. Also, the costs to be properly taken into account are not pecuniary costs alone but opportunity costs as defined in Section ID because the design options will often affect other parameters than reliability in the systems effectiveness relationship. This is another way of saying reliability is not the ultimate MOE. However, these complexities notwithstanding, the allocation could in principle be solved for an optimal decision of obtaining given system reliability at least cost. In the solution set of all subsystem

reliabilities we would expect the marginal product per dollar to be equal for every input - that is, the marginal unit of reliability to cost the same in every subsystem.

In practice the reliability goals may not be set in this manner and possibly with good reason, considering all the uncertainties in any calculation. However, conducting the cost estimating exercise may give some valuable insights into parameter sensitivities and be of great assistance in making good judgments on alternatives, even though little faith is put in exact attainment of the calculated values. There may also be good reason to make decisions about the re-entry subsystem design independently of the other subsystems - except for using the cost of the over-all missile system as a measure for pricing the product, the reliability provided within the REB subsystem. On technical grounds the REB is from a functional view-point relatively independent of the rest of the missile functions. Most of the fallibility occurs in a device which operates during time when the REB has been separated from the missile. Also, techno-logical advances or reaction to enemy defenses may dictate putting new re-entry systems on the same missile. Lastly, there are institutional impediments to close engineering optimization in the development of the re-entry body with the rest of the missile because the warhead portion of the REB is provided by the Atomic Energy Commission, a co-partner with the Department of Defense in REB design changes. Negotiating interface changes with an autonomous partner can be a lengthy process. This impediment may seem unreasonable but it is real.

Due to the high cost of a strategic missile system for delivery of the weapons, it soon becomes apparent that the reliability provided

in the REB should be relatively high, well into the area of decreasing marginal returns of reliability gained from improvement activities. There is then a temptation to merely estimate what might be technically feasible without regard to cost and set this as the reliability goal. This could be the proper goal, but it might be too high. It might be cheaper to buy more missiles rather than trying to push REB reliability into a very high marginal cost region.

Consider the MOE for the next higher level criterion to be the capability of inflicting damage on a given target complex. The number of REB's required is a decreasing function of their reliability while the cost of the REB's is an increasing function of their reliability. For reasons cited above the over-all problem of minimizing the total cost of a system to do a given amount of damage will not be worked directly including the REB as one of the subsystems, but the marginal cost curve of units of REB reliability will be generated from various reliability levels. This curve is the supply curve of reliability and is useful in finding an optimal level of reliability if we can provide the proper marginal value of REB reliability consistent with the ultimate MOE.

If one assigned a constant price or value for one unit of reliability percentage[1], this price would be proportional to the fractional cost[2] of the missile system required to place one weapon on target

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[1]This assumes that the price of product is constant no matter what quantity of product is supplied, or, in other words, that all units of ΔR are of equal value to the customer.

[2]By fractional cost we mean the total cost of the missile system divided by the number of weapons that system will deploy. We wish to reserve the term average cost for total cost of reliability improvement effort per weapon divided by units of reliability improvement attained.

divided by 100%. For this purpose what is meant by cost of the missile system? It is certainly more than the initial investment cost of missiles and also more than the investment cost of missiles and the remainder of the system, such as the launch platform and maintenance facilities. The most suitable cost would seem to be the present value, referred to the start of some planning period, with all investment and operational costs of the entire missile system included over the expected life cycle and discounted at an appropriate interest rate. Less than 100% reliability can result in a weapon's failure to function any time during its life cycle. This manner of using present value to relate to price of product implicitly assumes constant probability of use in anger throughout the life cycle. This assumption may not be easily defended but is no less reasonable than any alternative assumption in this regard. The cardinal value is not too significant in the analysis for the following reasons. We will be comparing the price of product to the cost of reliability percentage at the margin. It seems logical that the reliability units at the margin will not be as valuable to the consumer as the reliability units in the lower range of reliability. First consider this matter if the mission were deterrence. Certainly anything above zero reliability is of some value and its unit value may increase as reliability improves to a significant level of capability. Eventually we expect marginal value to decrease, and the level needed for any given confidence of deterrence becomes very fuzzy.[1] Next, if

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[1]Of course, all that is really needed for deterrence is the enemy's belief in the destructive capability of the system. It has been said that if the missiles were not ready on time, the first POLARIS submarine should have gone to sea with telephone poles in the missile tubes. But in our society it would be difficult to convince the enemy we had an effective weapon system if we did not believe it ourselves.

the MOE were capability of destruction of given portions of the value of a target complex, in most targeting schemes we would still have diminishing returns as the number of weapons on target increases. The price of product based directly upon the fractional cost of the missile system may thus be looked upon as an upper bound of the value of the product as implicitly determined by the decisions of some higher levels of authority in making an investment in the system.[1]

This manner of prorating costs to each weapon in the case of a tactical missile system would result in estimates of price of product of considerably lower order of magnitude than for a strategic missile. This would not be due alone, or possibly not even primarily, to the lower cost of the tactical missile. A major difference is that many missiles may be used in each launch facility; that is, there is a capability to reload in the tactical systems while the strategic systems are only supported for a one-shot war. If many reloads are expected, the cost of the entire system is divided by a large number in prorating cost to each weapon. A common practice in cost effectiveness studies of assuming the military worth of a weapon equal to the cost of the weapon alone then is not too bad in this event. On the other hand, the costs of some of the activities to improve reliability for the tactical weapons may not be that far removed from similar costs on the strategic weapons, particularly in the category of direct costs as opposed to opportunity costs. Thus, we might expect some

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[1]This conclusion follows because the expectation of weapons which will detonate on target is a direct multiplicative factor of expected reliability. The value of product (ΔR) then, if calculated by pro rata costs of the entire system and treated as constant throughout the range of ΔR provided, would be too high at the margin of reliability improvement due to the decreasing utility of unit ΔR as reliability increases.

dramatically different conclusions on how far to pursue reliability improvement in the different systems.

B. Generalized Lagrange Optimization Problem

If we consider several different activities to improve reliability, the efficient combination of those activities is a constrained optimization problem:

Maximize X where $X = \Delta R$, the increase in reliability above some baseline design value; subject to $\sum C_i = C$ where C_i is the total cost of each activity, and the summation is over all activities.

Where the X under consideration is a small range of potential reliability improvement and the number of activities each with small individual payoff is limited, it may be reasonable to consider $X = \sum X_i$ where X_i is the ΔR from each activity, in spite of the interdependencies which exist over large ranges and being careful to throw out non-feasible solutions where the level of some activities might make the assumed cost versus payoff of other activities incorrect. The relation between each X_i and its corresponding C_i would be expected to be a set of numbers corresponding to different design options or program plans. The X_i is the expected reliability improvement from that activity level and C_i the expected cost, either pecuniary or opportunity. Instead of having only a monetary cost constraint, one could conceive of several other constraints, such as allowable accuracy degradation or weight allocation. Indeed, in the usual way of doing business these type of constraints are likely to be dictated to the development activity. But if the valid MOE is kill potential on target, assignment of an accuracy requirement,

for example, by higher authority implies that the suboptimization problem for weapon development activities has been solved. It appears more logical to translate all the constraints into pecuniary or opportunity costs in monetary terms and then look for the efficient solution for use of development activities.

Thus to solve the suboptimization problem of where to put the reliability improvement effort we have a problem with the following characteristics: Maximize a payoff function, reliability improvement X, made up of the sum of several semi-independent activities each producing an X_i at a cost C_i, with an over-all constraint of $\sum C_i = C$. Since we are not putting great faith in a particular quantitative value of an optimum solution, we want a technique which gives considerable insight into the qualitative nature of the parameters without unduly extensive computation. The technique suggested is that of the Generalized Lagrange Multiplier Method described in reference [13] for solving problems of optimum allocation of resources, which can be applied similarly here for optimum allocation of reliability improvement activities, each utilizing the one monetary resource.[1] For each activity we will maximize the unconstrained function

$$X_i - \mu C_i$$

where μ is the Lagrange multiplier for the over-all problem. We do this for a range of μ values. Then a set of X_i and C_i which is feasible

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[1] A more complete explanation of application of the Generalized Lagrange Multiplier Method to this problem is given in Appendix 1.

represents optimal solutions to the overall problem:

$$\text{Maximize: } X = \sum X_i$$

$$\text{Subject to} \sum C_i = C.$$

A plot of C versus X for various μ values will build up a total cost curve from which we can derive average and marginal cost cruves. As in the ordinary Lagrangean formulation, the μ corresponding to any solution point is $\frac{dX}{dC}$ or $\frac{1}{\mu}$ equals the marginal cost of the reliability improvement.

It should be emphasized that when speaking of average and marginal costs the quantity of product referred to is units of reliability (or units of hazard product as defined in section IB). In normalizing these costs to be applicable to one weapon, of course, some calculations are carried out in the usual sense of a fixed cost for a certain development activity and unit cost per weapon for the same activity. Thus our C_i (a total cost in terms of product X_i) when a pecuniary cost is actually a sum of (1) a per weapon cost and (2) the quotient of a fixed cost divided by the number of weapons proposed for total production. It is clear that opportunity costs as used here must be calculated so as to be applicable to one weapon deployment capability in the fleet in order that they may be added directly to the normalized pecuniary costs (neglecting a small amount of spares which would bring production totals above deployment capability).

Consider a hypothetical program where the weapon is an REB for a strategic missile. The warhead in the REB will be caused to detonate at a prescribed height above target by an electronic arming and fuzing device (AFD) in the REB. We will assume that all the

fallibility (reliability reduction) comes from the AFD. Let us assume we want to decide upon design alternatives in four activity areas of reliability improvement to a basic design which presently exists with an expected initial reliability of R_0 resulting therefrom. Expected reliability improvement and cost estimates for different levels of intensity are needed for each activity in order to tabulate the reliability improvement (X_i) versus cost (C_i) relationships for use in a generalized Lagrangean calculation. The activities chosen and an approach for generating the X_i versus C_i relationships are described in Appendix 2.

C. Generalized Lagrangean Calculation

To illustrate use of the generalized Lagrangean we will assign some numbers to our hypothetical missile development program.

Assume a missile system whose projected present value of cost exclusive of provision of the REB's at the beginning of a ten-year planning period is 20 billion dollars. The system has launch facilities for 500 missiles, which pro rates 40 million dollars per missile, neglecting consideration of spares. If each missile carries ten REB's, this pro rates four million dollars per REB, and assuming one million dollars for the cost of the basic REB, we assign a value to deployment of each REB of five million dollars. Calling this the worth of close to 100% reliability, we assign a price of ΔR as 50 thousand dollars per one percent of ΔR.

Assume the basic REB design has an expected reliability of 90% and the ($X_i = \Delta R$) versus C_i relationships obtainable for activities i = 1 to 4 are estimated to be that shown in Table I. The activity

numbers refer to those described in Appendix 2. We will keep in mind an additional constraint that a solution which shows $\sum_{i=1}^{3} X_i > 6$ is not feasible, the reason being that activity #4 could not produce reliability improvements at costs given due to decreasing marginal returns if this high an aggregate intensity level for the first three were used.

i	X_i (%)	C_i (K$)
1	1	25
	2	100
	3	300
2	1	10
	2	50
	3	200
3	1	20
	2	70
	3	300
4	1	10
	2	40
	3	100

i = activity index number

X_i = reliability improvement

C_i = cost of reliability improvement

TABLE I. RELIABILITY IMPROVEMENT VERSUS COST

Next we use various values of μ and compute $X_i - \mu C_i$. Results are shown in Table II. The asterisk values in the table trace out the unconstrained maximizations for each activity level. The X_i and C_i corresponding to these values are added to give the total X and C for each value of μ. Average and marginal costs are then calculated as shown and plotted in Figure 1.

Note the plot shows that if we continue to price the product at 50K$ and use marginal cost, we will choose a program with an expected reliability of 96 or 97 percent. However, if average costs are used, which may be done in effect even though a conscious attempt at pricing the product is not done, the cost curves indicate one should push beyond the 99% level.

		μ					
i	X_i	.01	.011	.02	.025	.03	.04
1	1	.75	.725	.5*	.625	.25*	0*
	2	1.0*	.90*	0	-.50	-1.0	-2.0
	3	0	-.30	-3.0	-4.5	-6.0	-9.0
2	1	.9	.89	.8	.75	.7*	.6*
	2	1.5*	1.45*	1.0*	.75*	.5	0
	3	1.0	.80	-1.0	-2.0	-3.0	-5.0
3	1	.8	.78	.6	.5*	.4*	.2*
	2	1.3*	1.23*	.6*	.25	- .1	- .8
	3	0	-.30	-3.0	-4.5	-6.0	9.0
4	1	.9	.89	.8	.75	.7	.6*
	2	1.6	1.56*	1.2*	1.0*	.8*	.4
	3	1.6*	1.46	.2	-.5	-1.2	-2.6
$\sum_{i=1}^{3} X_i$		6	6	5	4	3	3
$\sum_{i=1}^{4} X_i = X$		9	8	7	6	5	4
$\sum_{i=1}^{4} C_i = C$		360	260	185	135	95	65
$AC = \frac{C}{X}$		40	32.5	26.4	22.5	19.0	16.3
$MC = \frac{\Delta C}{\Delta X}$			100	75	50	40	30

X = reliability improvement

C = cost for reliability improvement

AC = average cost

MC = marginal cost

μ = Lagrange multiplier

TABLE II. $(X_i - \mu C_i)$ FOR VARIOUS VALUES OF μ AND COST CALCULATIONS

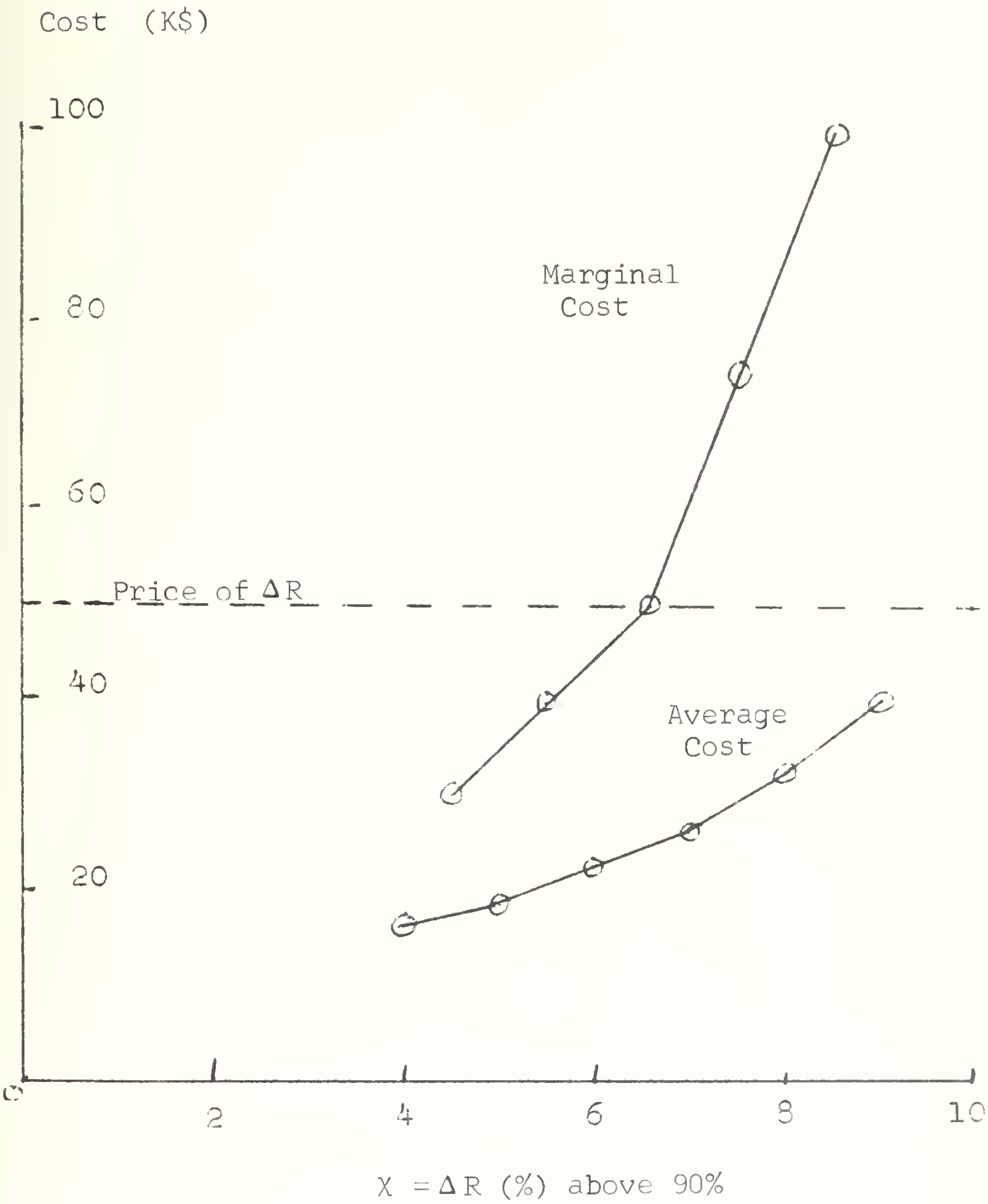

FIGURE 1. AVERAGE AND MARGINAL COST FOR RELIABILITY IMPROVEMENT

III. MAINTENANCE OF RELIABILITY

A. Reliability Model

In this chapter we will consider the problem of selecting an optimum design alternative, taking into account the degradation during dormant conditions. We will be looking for a minimum cost solution to obtain a given effectiveness level as a function of (1) the initial reliability and (2) the planned frequency of periodic tests to find and correct failures rates which occur during dormancy.

For an example of how dormant failure rates might be important, we will consider a simplified FTS for the REB of a hypothetical ballistic missile shown in Figure 2. There is an electronic AFD of given series complexity in the REB. The AFD is dormant until it rides a missile after aging for time t. It is activated after time t_2 of missile flight and must operate for time t_3 during re-entry before sensing the target and detonating the warhead. The expected hazard products during each phase are shown at the top of each box. Assume failure rates and times shown below:

Rate (failures per hour)	Time
$\lambda_1 = 2 \cdot 10^{-5}$	t = 10 years ($8.76 \cdot 10^{4}$ hours)
$\lambda_2 = .05$	t_2 = 2 minutes ($3.34 \cdot 10^{-2}$ hours)
$\lambda_3 = 1.0$	t_3 = 1 minute ($1.67 \cdot 10^{-2}$ hours)

Then, the expected hazard products and corresponding reliabilities are as shown in Table III. The reliability when new, $R_0 = e^{-\Sigma(\lambda_2 t_2 + \lambda_3 t_3)}$,

drops from 98.1% to 17.1% at the time of firing. If λ_1 were a factor of 10 lower, the resultant reliability would be 82.5% or a reduction of about 16% from R_0.

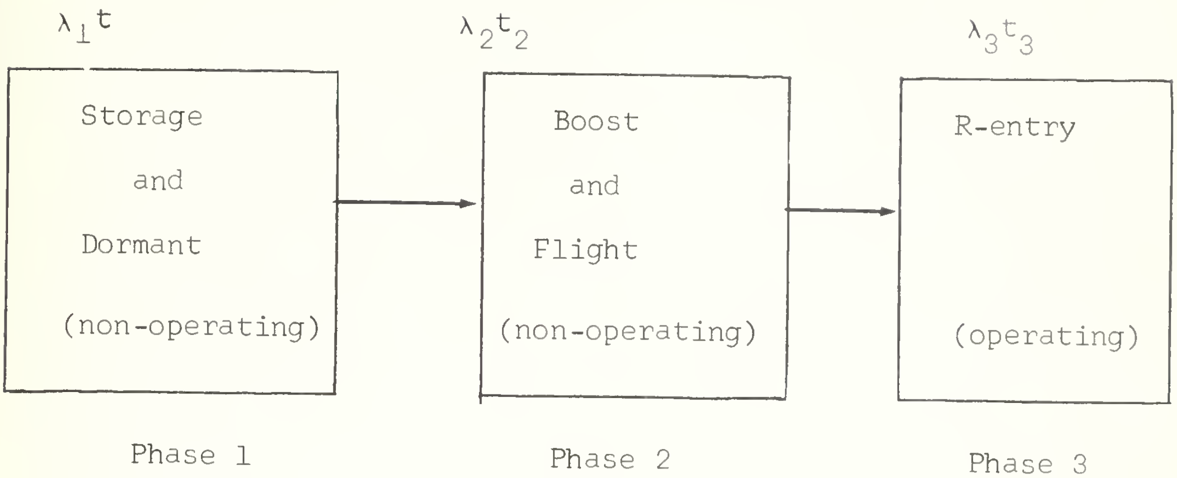

FIGURE 2. FACTORY-TO-TARGET SEQUENCE (FTS)

	$\lambda_i t_i$	R_i
$\lambda_1 t$	1.75	.174
$\lambda_2 t_2$	.00167	.998
$\lambda_3 t_3$	.0167	.983
$\sum \lambda_i t_i$	1.768	.171

λ_i = failure rate in failures per hour

t_i = time in hours

$\lambda_i t_i$ = hazard product

$R_i = e^{-\lambda_i t_i}$

TABLE III. HAZARD PRODUCTS AND RELIABILITIES

If a constraint were imposed that reliability must not be below 90% ($R \geq R^* = .9$) during a service life of ten years, under the wooden bomb philosophy the constraint would be violated. A no-test solution would be complete replacement of AFD's corresponding to their age when reliability drops to the constraining level. This is likely to be a very high cost solution. Another approach involving no tests might be feasible in a case where R_0 of the basic design and R^* are lower, allowing room for improvement in reliability of the basic design to increase the gap between R_0 and R^* enough to allow for degradation with age. In addition to monetary costs of a higher reliability design we would have degradation in the performance portion of the MOE due to the larger weight and volume of the AFD. This solution may in many cases be technically impossible because of the limited size of the gap between R^* and 100%, and the difficulty of increasing R_0 as it approaches 100%.

To consider how often periodic testing might be economically efficient to cull out failures, one must first determine whether or not reliability improvement is even feasible from a test and maintenance cycle. This has been analyzed for missile reliability, not considering degradation with age but from the viewpoint of raising the reliability of a lot of missiles by test and maintenance [14]. The model has the following parameters:

α = probability of a reliable missile being rejected by test (producer's risk)

β = probability of an unreliable missile being accepted by test (consumer's risk)

γ = degradation of missile reliability during test and maintenance cycle

P_R = probability that a missile will be repaired during maintenance cycle

Equations are derived for the new reliability of a lot for given parameters. For example, using values of

α = .10

β = .20

γ = 7%

P_R = .90

the author found that for initial reliability of a lot greater than .892, the result of test and maintenance is a worsening of the reliability of the lot tested. He then defines efficiency as $\frac{\Delta R_0}{R_0}$ and effective maintenance cost as maintenance cost divided by efficiency, and shows that effective maintenance cost doubles in the region of R_0 from 80 to 85%, and of course becomes infinite near 89% for the above parameter values.

Let us assume that for our REB maintenance cycle we have condensed these parameters into a factor, "a", which is the fraction of failures discovered and corrected, assuming negligible degradation during test.[1] Then to the FTS we add a repair/replace loop as shown in Figure 3.

For

a = .9

t_4 = 9½ years (dormant time before test)

t_5 = ½ year, (dormant time after test)

we get the results shown in Table IV for the two λ_1 values considered earlier. The reliability after ten years without testing compared

[1] If not assuming negligible degradation during test we could still define an "a" such that (1 - a) times the original number of failures is the number of failures in the lot after the maintenance cycle.

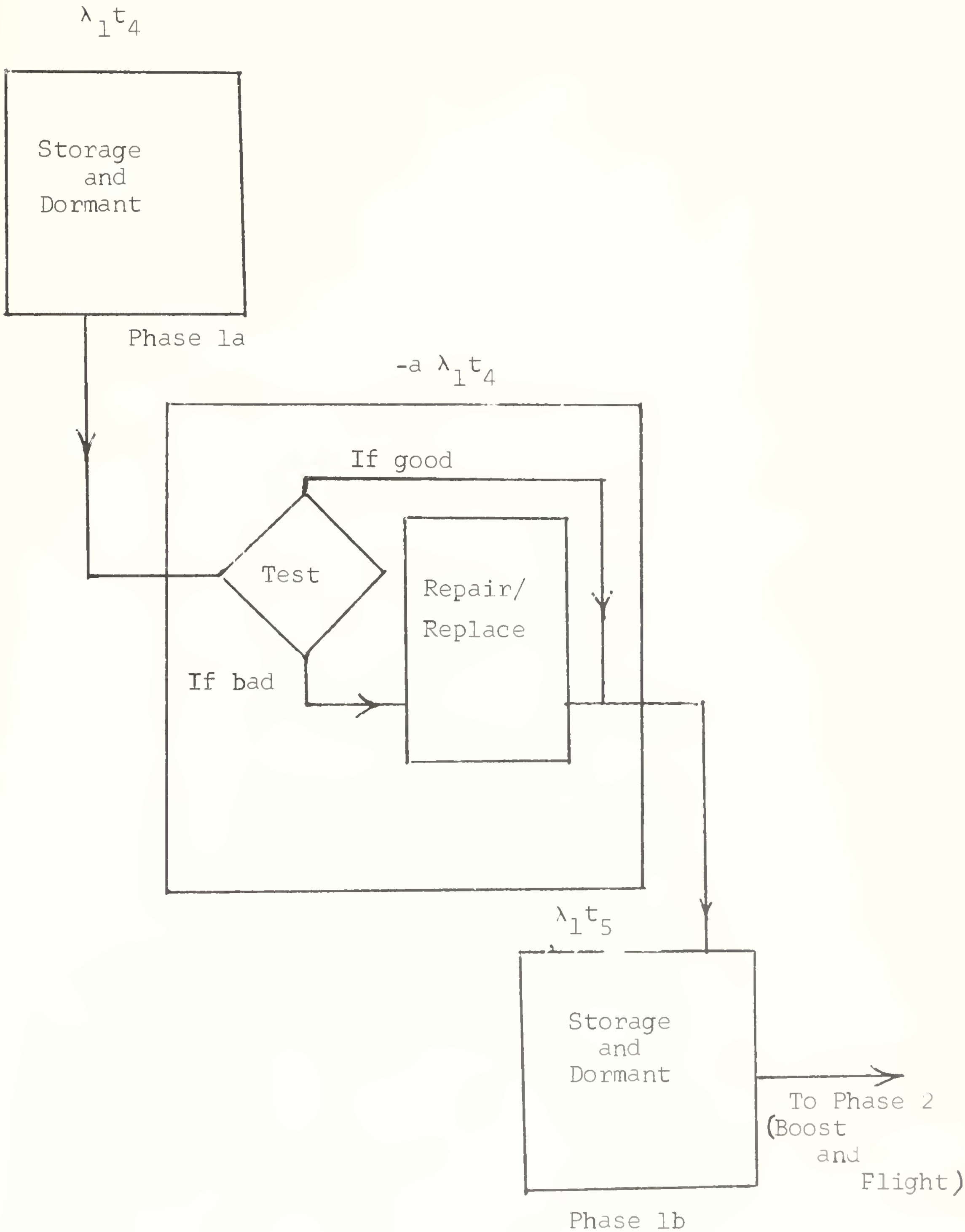

FIGURE 3. FTS WITH TEST AND REPAIR/REPLACE LOOP

	$\lambda_1 = 2 \cdot 10^{-5}$		$\lambda_1 = 2 \cdot 10^{-6}$	
	$\lambda_i t_i$	R_i	$\lambda_i t_i$	R_i
$(1 - a)\lambda_1 t_4$	.166	.847	.0166	.9835
$\lambda_1 t_5$	.0876	.916	.00876	.9913
$\lambda_2 t_2$	.00167	.998	.00167	.998
$\lambda_3 t_3$	.0167	.983	.0167	.983
$\sum \lambda_i t_i$	.272	.762	.0437	.957

λ_1 = dormant failure rate (failures per hour)

a = fraction of failures corrected in maintenance cycle

$(1 - a)\lambda_1 t_4$ = hazard product after dormant time t_4 and maintenance cycle

$\lambda_1 t_5$ = hazard product after test cycle during dormant time t_5

$\lambda_2 t_2$ = hazard product during boost

$\lambda_3 t_3$ = hazard product during re-entry

$R_i = e^{-\lambda_i t_i}$

TABLE IV. RESULTS OF MAINTENANCE CYCLE

to the case with testing for the two dormant failure rates is:

17.1% raised to 76.2%

82.5% raised to 95.7%.

B. Problem Analysis for Absolute Minimum Reliability Requirement

For simplification of cost analysis let us condense the FTS even more and assume that all failures occur during dormancy time before missile flight and operating time during re-entry.

Then, at time t_1 the hazard product

$$h_{t_1} = \lambda_1 t_1$$

and after test and repair/replace the hazard product

$$h'_{t_1} = (1 - a)\,\lambda_1 t_1$$

For periods of equal length t_1 between testing, after n periods:[1]

$$h_{nt_1} = \left[(1-a)^{n-1} + (1-a)^{n-2} + \ldots .(1-a) + 1 \right] \lambda_1 t_1$$

$$h'_{nt_1} = \left[(1-a)^{n} + (1-a)^{n-1} + \ldots .(1-a) \right] \lambda_1 t_1$$

---

[1]These two expressions are derived as follows. After each period of length t_1, $\lambda_1 t_1$ is added to the hazard product. After each maintenance cycle the resultant hazard product is obtained by multiplying by (1 - a). For example, for n = 2:

$$h_{2t_1} = h_{t_1} + \lambda_1 t_1 = (1 - a)\lambda_1 t_1 + \lambda_1 t_1 = \left[(1 - a) + 1\right] \lambda_1 t_1$$

$$h'_{2t_1} = (1 - a)\, h_{2t_1} = \left[(1 - a)^2 + (1 - a)\right] \lambda_1 t_1$$

Neglecting terms of (1 - a) raised to power of two or more,

$$h_{nt_1} \cong (2 - a)\, \lambda_1 t_1$$

$$h'_{nt_1} \cong (1 - a)\, \lambda_1 t_1 = h'_{t_1}$$

Then for some selection of t_1 in combination with a given initial reliability R_0 corresponding to $R_0 = e^{-h_0}$ where $h_0 = \lambda_3 t_3$ is called the initial hazard product, the total hazard product h versus time over a service life of length $T = nt_1$ would look like Figure 4.

Let us examine a solution for the case where higher authority prescribes an absolute minimum reliability level as a requirement. This may not be the most logical way to consider the allocation of reliability effort, but it may often be the actual conditions under which it must be done. For many institutional and administrative reasons what should be treated as a first approach reliability goal may become a firm requirement. This would imply that the higher authority knows in absolute terms the maximum effectiveness necessary to achieve a certain result - such as a given kill capability to assure deterrence.

Suppose a constraint is set that each REB never have a total hazard product greater than h^* ($R \geq e^{-h^*}$). Let us assume that the REB has a basic h_0 which will not meet the constraint as a wooden bomb throughout its service life, but improvement in initial reliability is possible, and correction of dormant failures through test and maintenance is feasible. The cost function C_T for each REB will be made up of three parts:

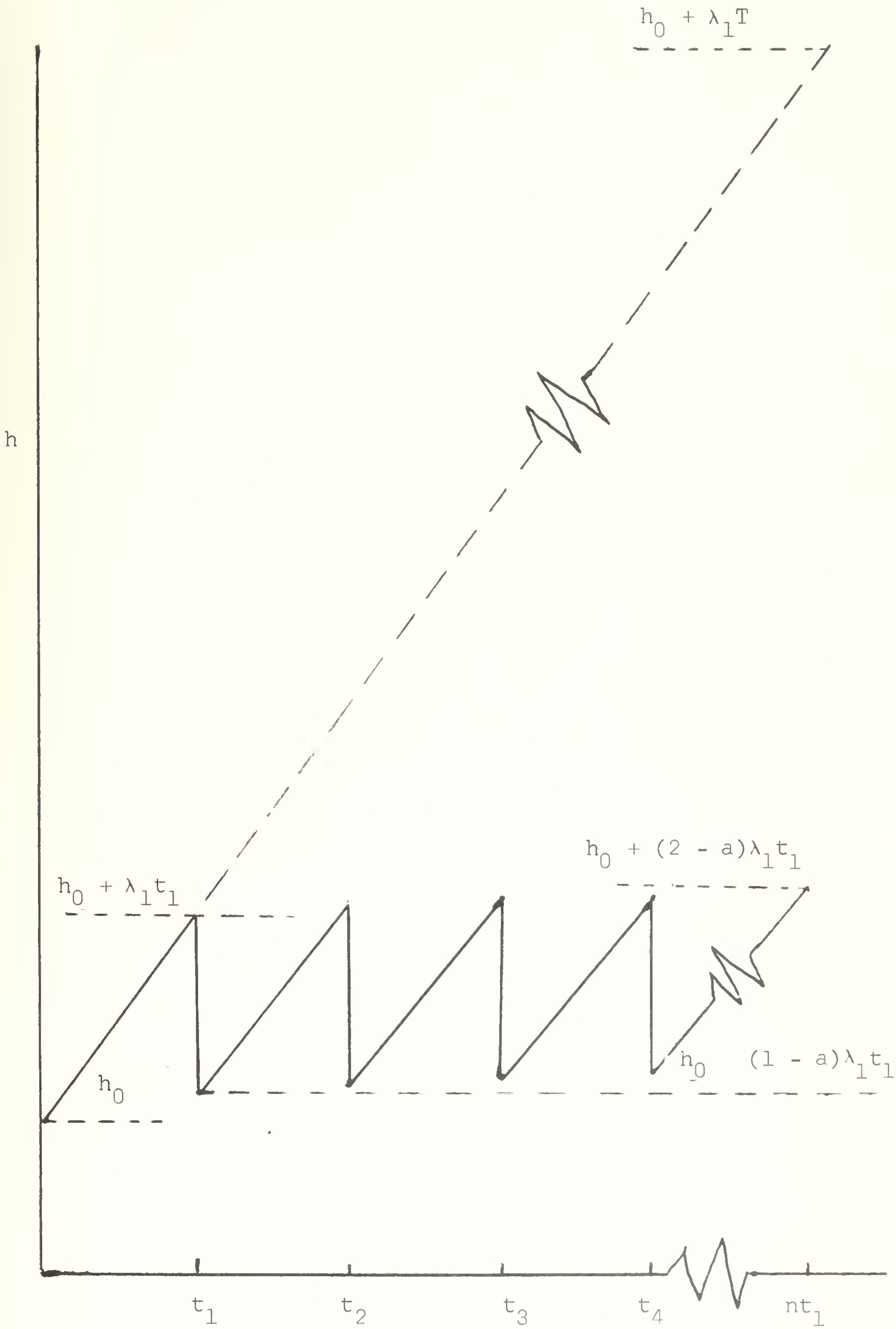

FIGURE 4. HAZARD PRODUCT VERSUS TIME

(1) $C(h_0)$ = the cost of obtaining a given initial reliability as computed by the generalized Lagrangean technique.

(2) Cost of testing, $\frac{C_m T}{t_1}$, where

C_m = cost of a test (monitoring) operation

T = prescribed service life

(3) Expected cost of repair or replacement for failures detected, $C_r \cdot (a\ \lambda_1 T)$, where

C_r = average cost of a repair operation.

We will consider the third term as a fixed cost, not a function of h_0 and t_1.[1]

We also assume "a" and λ_1 are not functions of h_0.[2]

---

[1]To be plausible for C_r, this assumes ability to repair the AFD. C_r would be a strong function of h_0 if replacement were made. But even complete replacement might be much smaller than $C(h_0)$ which is dominated by opportunity costs. If the AFD could not be replaced so that C_r was on the order of a complete REB cost, then the term probably becomes of more significance relative to $C(h_0)$.

[2]The assumption would be totally unrealistic for λ_1 if the "basic" reliability were of low order. For then the measures taken to improve h_0 in the area of manufacturing and quality control would be assumed to reduce dormant failures, which are due largely to built-in defects. However, on the assumption that some of these built-in defects cannot be detected by testing, the improved screening process will not discover them, and it is likely that the process cannot be improved to eliminate a defect whose nature is not known. In some regime of high initial reliability, λ_1 may be almost independent of h_0. If we assumed that u_1 would remain approximately a constant proportion of u_3, which is indicated by statistical results in some reliability regimes, we could represent λ_1 as a constant times h_0. The following analysis could be carried out on this basis, but the results would be much more complex to analyze. Note also that C_r would be expected to be a decreasing function of h_0 while λ_1 would increase with h_0. Thus the changes due to h_0 would tend to cancel each other.

The problem then is to minimize:

$$C_T(h_0, t_1) = C(h_0) + \frac{C_m T}{t_1} + C_r \cdot (a\lambda_1 T)$$

subject to:

$$h_0 + (2 - a) \cdot \lambda_1 t_1 \leq h^*.$$

Note that for a given h_0 the constraint equation determines t_1 because the largest t_1 possible will minimize cost for the given h_0. That is, the equality will hold in the constraint equation at an optimal solution. We can thus compute total costs for each of the h_0 values corresponding to design alternatives for which $C(h_0)$ values were computed by the generalized Lagrangean. Results of such a computation are shown in Table V for the following parameters:

C_m = 1 (K\$)

T = $8.76 \cdot 10^4$ hours

a = .9

λ_1 = 2.10^{-5} per hour

C_r = unassigned because the fixed cost does not affect the optimization

h^* = .10 and .06 (R^* = 90% and 94% respectively)

Variable Cost (VC) $= C(h_0) + \dfrac{C_m T}{t_1}$.

Note that for R^* = 90% no minimum cost is found in the regime of design alternatives considered although the values seem to be quickly approaching a minimum near the lowest value of R_0 equal 94%. In this case we might guess that there is something inconsistent in the value judgments which resulted in spending the total amounts indicated for

R_0 (%)	h_0	$C(h_0)$ (K$)	t_1 (10^3 hours)	$\frac{C_m T}{t_1}$ (K$)	VC (K$)
			$h^* = .10$ ($R^* = 90\%$)		
99	.01	360	4.10	21.4	381
98	.02	260	3.64	24.1	284
97	.03	185	3.17	27.6	213
96	.04	135	2.72	32.2	167
95	.05	95	2.27	38.6	133
94	.06	65	1.82	98.2	113 ←
			$h^* = .06$ ($R^* = 94\%$)		
99			2.27	38.6	399
98			1.82	48.2	308
97			1.36	64.3	249
96			.910	96.5	232 ←
95			.455	193	288
94			0		

h^* = maximum hazard product allowable

$R^* = e^{-h^*}$

R_0 = initial reliability $= e^{-h_0}$

t_1 = period between tests

$C(h_0)$ = cost of initial reliability

$\frac{C_m T}{t_1}$ = cost of testing

VC = variable cost $= C(h_0) + \frac{C_m T}{t_1}$

TABLE V. TABULATION TO FIND MINIMUM VARIABLE COST AS A FUNCTION OF h_0 AND t_1

the total program but would then set R^* as low as 90%. For $R^* = 94\%$ the minimum cost is found at $R_0 = 96\%$. In summary, the solutions are:

	$R^* = 90\%$	$R^* = 94\%$
R_0	$< 94\%$	96%
t_1 (hours)	$< 1.82 \cdot 10^3$	$.910 \cdot 10^3$
n	≥ 48	96

Of course, if we examine the technical and operational constraints, an indicated optimum solution may not be feasible.[1]

Next let us pose the cost minimization problem for solution by use of a Lagrangean with the constraint expressed as an equality. As will be indicated, it is not particularly useful in this case as a method of finding the optimal solution but, as pointed out by Samuelson in reference [15], it is often possible by using this framework to determine the qualitative behavior of solution values in respect to change of parameters.

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[1]For example, the solution for the $R^* = 94\%$ case calls for a monitoring operation approximately every 5½ weeks. The C_m assigned, if based only on the pecuniary cost of the monitoring operation, assumes the REB would be available for test anyway, presumably to coincide with down time on some other part of the weapon system. If this were not the case and C_m had to cover weapon unavailability costs, C_m would be much larger and more dominating in the solution. Also, we are assuming the number of tests over a service life is not large enough to cause the failure rate of any of the components to move into the wearout region. This is a function of the nature of the monitoring operation. We could reduce the thoroughness of the test to avoid wearout but would also inevitably reduce "a", the fraction of failures corrected which depends on the number detected. So in the end, if none of the solutions look feasible, we reconsider the basic design (perhaps allowing more for the characteristic that our devices be amenable to testing) and recompute the $C(h_0)$ relationships.

Let the Lagrangean,

$$C_L = C_T - \mu \cdot \left[h_0 + (2 - a) \cdot \lambda_1 t_1 - h^* \right]$$

First order conditions for an extremum:

$$\frac{\partial C_L}{\partial h_0} = \frac{\partial C_T}{\partial h_0} - \mu = 0 \qquad \text{or} \qquad \mu = \frac{\partial C_T}{\partial h_0} = \frac{\partial C}{\partial h_0}$$

$$\frac{\partial C_L}{\partial t_1} = \frac{\partial C_T}{\partial t_1} - (2 - a)\,\lambda_1 \mu = 0 \qquad \text{where } \frac{\partial C_T}{\partial t_1} = \frac{-C_m T}{t_1^{\,2}}$$

$$\frac{\partial C_L}{\partial \lambda_L} = - h_0 - (2 - a)\,\lambda_1 t_1 + h^* = 0$$

It follows[1] that $\mu = \frac{dC_T}{dh^*}$ which is the marginal imputed cost of the constraint.

---

[1]This is shown by taking total derivative of C_T and the constraint. and using first order conditions:

$$dC_T = \frac{\partial C_T}{\partial h_0}\, dh_0 + \frac{\partial C_T}{\partial t_1}\, dt_1 = \mu\, dh_0 + (2 - a)\,\lambda_1 \mu\, dt_1 =$$

$$\mu(dh_0 + (2 - a)\,\lambda_1\, dt_1)$$

$$dh^* = dh_0 + (2 - a)\,\lambda_1 dt_1$$

Then,

$$\frac{dC_T}{dh^*} = \mu$$

To solve for h_0 and t_1, note that by using the last equation we can express t_1 as a function of h_0. Then if we could express $\frac{\partial C}{\partial h_0}$ as a function of h_0, we could write the second equation with h_0 as the only unknown and solve for h_0. However, even if $\frac{\partial C}{\partial h_0}$ were linear in h_0 (which in general it is not), we would have a third degree equation to solve for h_0. In our discontinuous example, $\frac{\partial C}{\partial h_0}$ is approximated by straight lines, but, to take advantage of this, one would have to guess which segment would contain the solution point, and, if the solution is near a change in slope, the answer might be erroneous. Thus, the iterative tabular method seems the more practical for getting an indication of the approximate optimum solution in this case, but it depends on the form of the cost information.

Taking the total differential of the first order conditions gives the following matrix equation:

$$\left[\begin{array}{cc:c} \dfrac{\partial^2 C_T}{\partial h_0^2} & 0 & -1 \\ 0 & \dfrac{2C_m T}{t_1^3} & -(2-a)\lambda_1 \\ \hdashline -1 & -(2-a)\lambda_1 & 0 \end{array}\right] \cdot \begin{bmatrix} dh_0 \\ dt_1 \\ d\mu \end{bmatrix} = \left[\begin{array}{c} 0 \\ \hdashline \dfrac{T}{t_1^2}\, dC_m + \dfrac{C_m}{t_1^2}\, dT \\ + (2-a)\mu\, d\lambda_1 - \lambda_1 \mu\, da \\ \hdashline (2-a)\, t_1 d\lambda_1 - \lambda_1 t_1 da \\ - dh^* \end{array}\right]$$

This is of form $A y = b$ where

A is the bordered Hessian (the Hessian above and to left of dotted line) which must be positive definite for a minimum, a condition which is satisfied if $\frac{\partial^2 C}{\partial h_0^2}$ is positive[1], which is true over the relevant range of h_0 as depicted in Figure 5, which is plotted from the information in Table II.

Forming $y = A^{-1} b$

allows the qualitative examination of relations between parameters. For example, if $dT = d\lambda_1 = da = dh^* = 0$,

$$\frac{\partial t_1}{\partial C_m} = \frac{T}{t_1^2} \cdot d_{22} \qquad \text{where}$$

d_{22} refers to the second diagonal element of A^{-1} which is known to be positive. This shows how the period between tests increases with the cost of tests.

To illustrate further use of the second order conditions, let us put in the values for our solution for the $h^* = .06$ case. The values are summarized below:

---

[1]The sign of the determinant must be negative because there is one constraint. Expanding by the first column gives

$$\frac{\partial^2 C_T}{\partial h_0^2} \cdot \left(- (2 - a)^2 \lambda_1^2\right) - \frac{2C_m T}{t_1^3} \,.$$

Thus both terms are negative if $\frac{\partial^2 C_T}{\partial h_0^2}$ is positive.

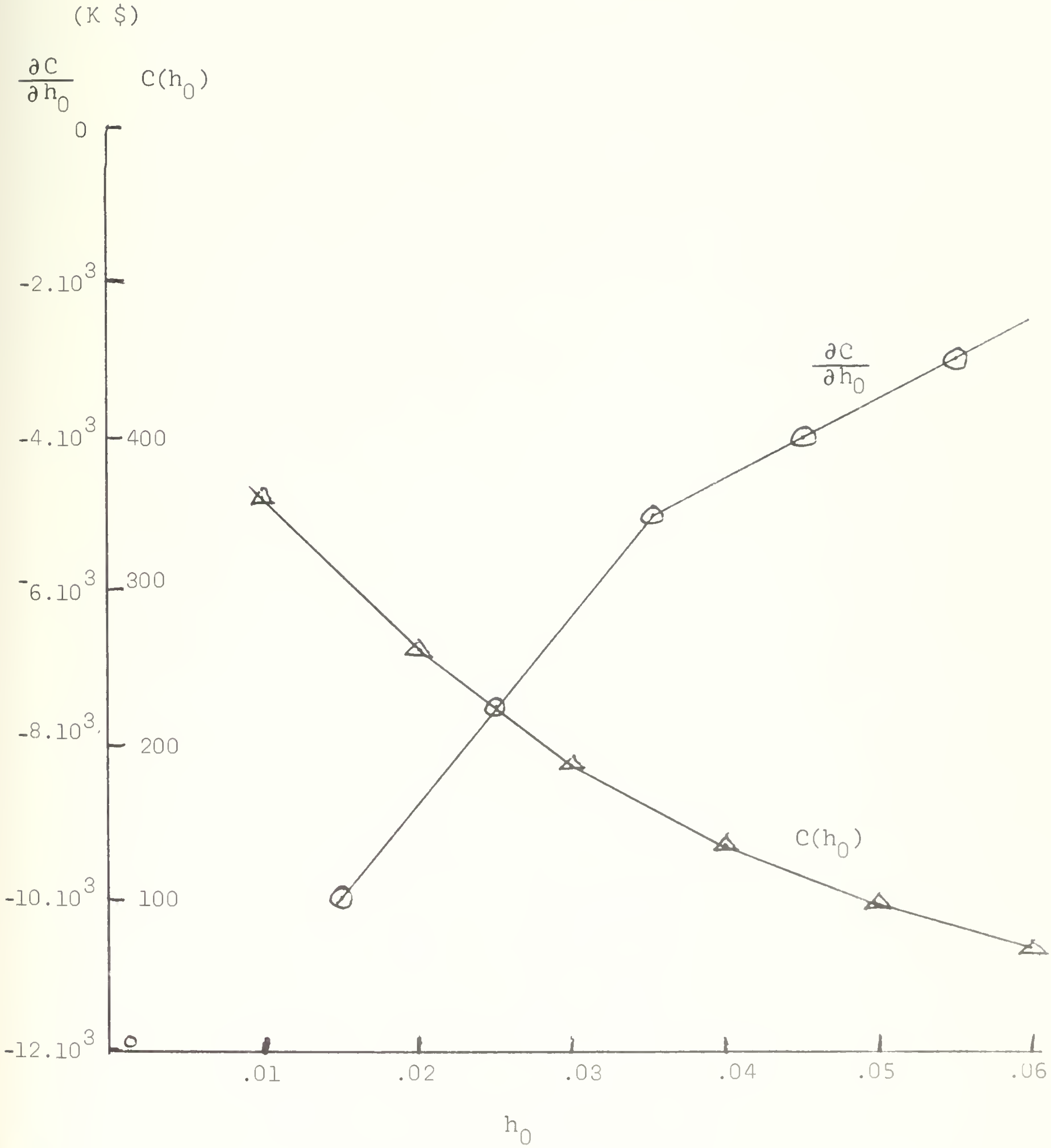

FIGURE 5. COST AND MARGINAL COST VERSUS INITIAL HAZARD PRODUCT

Solution Values

$h_0 = .04$

$t_1 = 910$ hours

$n = 96$

$C_T = 135 + 96 + 1.58\ C_r$

$\quad = 231 + 1.58\ C_r$

Parameters

$T = 8.76 \cdot 10^4$ hours

$\lambda_1 = 2 \cdot 10^{-5}$

$a = .9$

$C_m = 1$ K$

$h^* = .06$

Using these values the matrix equation after multiplying by A^{-1} becomes:

$$\begin{bmatrix} dh_o \\ dt_1 \\ d\mu \end{bmatrix} = \begin{bmatrix} 1.72 \cdot 10^{-6} & -7.83 \cdot 10^{-2} & -.83 \\ -7.83 \cdot 10^{-2} & 3.56 \cdot 10^3 & -7.83 \cdot 10^3 \\ -.83 & -7.83 \cdot 10^3 & -8.3 \cdot 10^4 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{0}{.106\ dC_m + 1.21 \cdot 10^{-6} dT} \\ \dfrac{-4.95 \cdot 10^3 d\lambda_1 + 9.0 \cdot 10^{-2} da}{10^3 d\lambda_1 - 1.82 \cdot 10^{-2} da} \\ -dh^* \end{bmatrix}$$

We can then find $\dfrac{\partial t_1}{\partial C_m} = 377$ and deduce that t_1 will increase with C_m but proportionally less rapidly by about 1/3 from the equilibrium position.

Since dormant failure rate may be a parameter with very large proportional uncertainty, let us examine the effect of its change with all other parameters constant. The matrix equation shows

$$\frac{\partial h_0}{\partial \lambda_1} = -442$$

$$\frac{\partial t_1}{\partial \lambda_1} = -2.54 \cdot 10^7$$

$$\frac{\partial \mu}{\partial \lambda_1} = -4.42 \cdot 10^7$$

Suppose $\Delta\lambda_1 = 10^{-5}$, corresponding to a 50% increase in failure rate. Then,

$$\Delta h_0 \cong -.004$$

$$\Delta t_1 \cong -254 \text{ hours}$$

$$\Delta \mu \cong -442 \text{ K\$ per unit h (greater in absolute value).}$$

The h_0 and μ solution values have a smaller proportional change (about 1/10) than t_1, which must be decreased by about ¼. This could be an encouraging result if t_1 could be changed easily during the operational life to take care of an unexpected dormant failure rate. But it may not be the case and indeed, in this example, it was indicated likely that the optimal solution initially called for would probably not be feasible because of t_1 being too short.

These results suggest that it may be most practical in many systems to work the problem sequentially - first, finding an efficient solution for initial reliability allowing for the possibility of unfavorable results in the dormant failure rate area; later maximizing the return over time, delaying the details of the monitoring program as long as possible in order to utilize the best predictive information that can be gleaned from the program before making the decision on the nature of the monitoring program.

C. Problem Analysis for Average Reliability Over Time

What is proposed as a more logical approach than the minimum reliability requirement of the previous section is to investigate what average reliability level over time can be attained. This seems better for several reasons. First, the likelihood of use is a completely

random event for which average values over time are appropriate if the swings above and below the average are not too large. It appears that our answers tend to point toward a fairly larger number of test cycles which will keep the variance small. Secondly, the variations in reliability plotted over time are for a single weapon. In a practical case the condition of different groups of weapons in the fleet being in different parts of the cycle will cause a smoothing out of the average reliability over time when looking at the fleet as a whole. And it is this average as a whole which determines the effectiveness of the entire weapon system.[1]

Another way of looking at the objective is to consider that the goal is to maximize the proportion of time which the weapon is good (no dormant failures before firing). This viewpoint allows closer analogy to investigation on inspection programs for randomly failing equipment in systems which are not one-time use but operate more or less continually. For example, the paper by Weiss [16] which refers to some of the many earlier papers on the subject says that the objective of any checkout program is to minimize the down time due to system failure (or maximize the expected operational readiness of the system); and that when the reliability function is negative exponential, the optimal checkout policy must be a periodic one. It is not considered important here to rigorously show that the optimal inspection program is periodic because practical considerations would undoubtedly dictate

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[1]Of course, much of this argument is valid for using the minimum reliability requirement also, but the concept of adjusting effectiveness levels dependent on the cost of attainment seems more palatable when we get away from the rigid specification of a minimum requirement. Also, the minimum level framework may tend to give answers which pay too much for avoidance of dropping below the minimum level for a very short period of time.

its use, the advantage of simplicity in scheduling the checkouts outweighing any small theoretical advantage of a more complex program.

Again considering the problem in terms of hazard product and looking back to Figure 4, we see that without testing over the service life T, the expectation of hazard product,

$$\bar{h}_{w.o.} = \frac{h_0 + (h_0 + \lambda_1 T)}{2} = h_0 + \frac{\lambda_1 T}{2}$$

With testing,

$$\bar{h} \cong \frac{\left[h_0 + (1 - a)\lambda_1 t_1\right] + \left[h_0 + (2 - a)\lambda_1 t_1\right]}{2}$$

$$= h_0 + \lambda_1 \left(\frac{3-2a}{2}\right) t_1$$

or defining $b = \dfrac{3-2a}{2}$

$$\bar{h} \cong h_0 + b\,\lambda_1 t_1$$

Then the improvement (decrease) in $\bar{h}$ is represented by

$$\bar{h}_{w.o.} - \bar{h} = \frac{\lambda_1}{2} (T - 2bt_1)$$

If we can assign a value P to a unit of decrease in h, then the "revenue" from the testing program is

$$\tfrac{1}{2}P\,\lambda_1 (T - 2bt_1).$$

As before, for a given h_0, we can represent the cost of the testing program as

$$C = \frac{C_m T}{t_1} + C_r \cdot a\,\lambda_1 T$$

We could then find t_1 to maximize "profit" expressed as revenue minus cost or $\frac{1}{2} P\lambda_1 (T - 2bt_1) - \left(\frac{C_m T}{t_1} + C_r \cdot a \lambda_1 T\right)$.

First order condition:

$$-bP\lambda_1 + \frac{C_m T}{t_1^2} = 0$$

$$\text{or} \quad t_1 = \sqrt{\frac{C_m T}{bP\lambda_1}}$$

Second derivative:

$$\frac{-2C_m T}{t_1^3}$$

verifies positive root corresponds to a maximum profit. For example, using the same parameter values as in previous section for the $h_0 = .04$ design, noting that $b = .6$ corresponds to $a = .9$, and assigning $P = 5 \cdot 10^3$ K\$, our upper limit of the value of unit h, corresponding to 50 K\$ per 1% of ΔR, results in

$$t_1 = 1.21 \cdot 10^3 \text{ hours} \qquad \text{and}$$

$$n = 72.5$$

or an inspection about once every 6.6 weeks.

For $\lambda_1 = 2 \cdot 10^{-6}$ per hour, the solution would be

$$t_1 = 3.82 \cdot 10^3 \text{ hours}$$

$$n = 23$$

or about once every five months. Thus, reducing the estimate of dormant failure rate by a factor of ten still leaves a solution which calls for considerable inspection.[1]

For an analysis of the parameters consider the problem in Lagrangean format, leaving out an assigned value of unit hazard product (P). The problem is to minimize

$$C = \frac{C_m T}{t_1} + C_r \cdot a\, \lambda_1 T$$

subject to

$$h_0 + b\lambda_1 t_1 = \bar{h}.$$

Note that once all parameters are assigned the constraint equation determines t_1 which then determines the cost so this is really not an extremum problem at all. However, using the Lagrangean format as in the previous section allows some insight into the nature of the solution with qualitative change of parameters.

The Lagrangean: $C_L = C - \mu\left[h_0 + b\lambda_1 t_1 - \bar{h}\right]$

First order conditions:

$$\frac{\partial C_L}{\partial t_1} = \frac{-C_m T}{t_1^{\,2}} - b\lambda_1 \mu = 0$$

$$\frac{\partial C_L}{\partial \lambda_L} = -h_0 - b\lambda_1 t_1 + \bar{h} = 0$$

---

[1]Of course, the maximum solution could be at a point of negative profit if the "fixed cost" term containing C_r were high enough. But this term is not expected to be dominating even if C_r involved complete replacement because of P being much larger than C_r.

Then

$$t_1 = \frac{\bar{h} - h_0}{b\lambda_1}$$

$$\mu = \frac{-C_m T}{b\lambda_1 t_1^{\,2}} = \frac{-C_m T b\lambda_1}{(\bar{h} - h_0)^2} = \frac{dC}{d\bar{h}}$$

Taking total differential of the first order conditions and multiplying through by inverse of Hessian as before gives the following matrix equation:

$$\begin{bmatrix} dt_1 \\ d\mu \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{b\lambda_1} \\ \dfrac{-1}{b\lambda_1} & -\dfrac{2C_m T}{b^2\lambda_1^{\,2} t_1^{\,3}} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{T}{t_1^{\,2}}\, dC_m + \dfrac{C_m}{t_1^{\,2}}\, dT + b\mu d\lambda_1 + \lambda_1 \mu da \\ bt_1 d\lambda_1 + \lambda_1 t_1 db + dh_0 - d\bar{h} \end{bmatrix}$$

To compare this approach with the (h_0, t_1) optimization case of previous section where we set $h^* = .06$ and found

$$h_0 = .04$$

$$t_1 = 910 \text{ hours},$$

our constraint equation indicates that we would set

$$\bar{h} = .0532$$

and $$\mu = -\,8.83 \cdot 10^3 \text{ K\$}.$$

This μ value is larger in absolute value than the value of product assigned (P) in our profit maximization calculation so the smaller frequency of checking in that calculation is consistent.

With these parameter values and for changes in λ_1 only:

$$\frac{\partial t_1}{\partial \lambda_1} = -4.55 \cdot 10^7$$

$$\frac{\partial \mu}{\partial \lambda_1} = -4.38 \cdot 10^8.$$

Then if $\Delta\lambda_1 = 10^{-5}$

$$\Delta t_1 \cong 455 \text{ hours}$$

$$\Delta \mu \cong -4.38 \cdot 10^3 \text{ K\$ per unit h (greater in absolute value)}$$

Compared to the (h_0, t_1) optimization case, the change in λ_1 has a much greater effect on the solution values, -455 compared to -254 hours for t_1 and -4380 compared to -422 K$ per unit h for $\Delta\mu$.

Similarly for changes in b only,

$$\frac{\partial t_1}{\partial b} = -908$$

$$\frac{\partial \mu}{\partial b} = -2.63 \cdot 10^4.$$

If $\Delta b = .1$, corresponding to $\Delta a = -.1$ (or $b = .7$ and $a = .8$),

$$\Delta t_1 \cong -90.8 \text{ hours}$$

$$\Delta \mu \cong -2.63 \cdot 10^3 \text{ k\$}.$$

The cost in the testing area per unit h is seen to change more rapidly than total costs. The combination solution for h_0 and t_1 tended to mask the importance of high "a" in keeping down costs.

The interdependency between some of the parameters is also well illustrated graphically. Looking at μ as a function of C_m and for the parameters chosen, $\mu = 8.83 \cdot 10^3\ C_m$ is plotted as the middle line in Figure 6. Choosing $\bar{h}$ so as to reduce $(\bar{h} - h_0)$ by half results in the lower line and doubling $(\bar{h} - h_0)$ yields the upper line. One could compare the intersection of the chosen line with the value of C_m against his idea of the value of the product, one unit of $\bar{h}$ reduction. Starting with the upper line, one would want to reduce $\bar{h}$ until the indicated μ or marginal cost equaled the value of the product. One can also see in this figure that, dependent on the choice of $(\bar{h} - h_0)$, marginal cost may or may not be very sensitive to C_m. For $(\bar{h} - h_0)$ very small (little degradation in reliability allowed) marginal cost is very sensitive to C_m.

This type of analysis could of course be extended to investigate the sensitivity of other parameters in a similar manner. Those parameters whose values are not well known and to which the solution is sensitive then provide valid warning against too much confidence in the indicated design solution.

Thus far we have indicated a method of economic analysis to maximize the value of expected reliability minus the cost of providing it. We will next look further at modifications based on utility and uncertainty considerations.

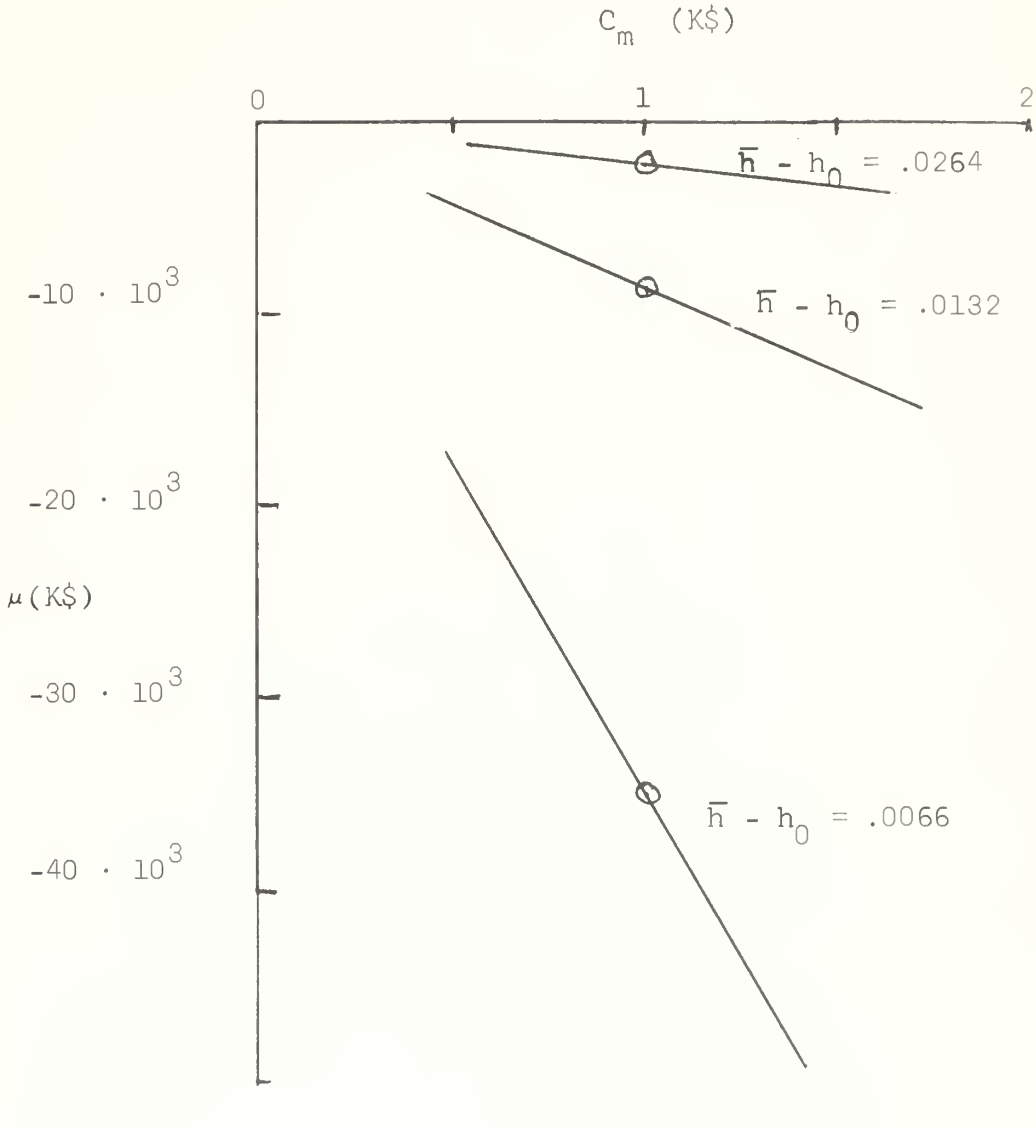

FIGURE 6. MARGINAL COST VERSUS COST OF MONITORING

IV. UNCERTAINTY CONSIDERATIONS

In this chapter we will consider in a qualitative way those activities of reliability improvement which were not a factor in any of the previous calculations. As indicated in Section ID the dividing line between risk based on objective probabilities and uncertainty evaluated by subjective judgment and probabilities is not clearcut. A descriptive activity might be categorized one way or the other, dependent on how much back-up data is available.

As one group of activities let us consider the reliability improvement processes we had numbered (1c), (2c), and (4b); namely,

(1c) Functional Design - Improvement of insensitivity to parameter changes

(2c) Packaging - Environmental stress protection

(4b) Operational Procedures - Mitigation of stress factors

These three activities, most particularly the last two, involve what stress factor, k, is to be assigned to the various phases of the FTS. The value of k in any event is one of the least well known factors in our calculations because the statistical accuracy of much of the data is based on experiments on components under standard conditions. How this data will apply to conditions on a yet-to-be-built weapon system is highly subjective. The insensitivity to parameter changes is more directly related to the question of tolerance failures versus catastrophic failures.

Again the piece part data is more clearly applied in the area of catastrophic failures of parts because the characteristics of the design of each device will determine whether a given change of parameters in a part will actually cause a failure of the device due to tolerance limits being exceeded. Another factor complicating calculation is the consideration of an environment caused by enemy countermeasures in which a device may have to operate. For example, the radiation environment from enemy defensive nuclear weapons will change the tolerance levels which determine whether an electronic device will still operate properly or not. Determination of this level is difficult enough in a benign environment. If one tries to design to be fairly immune to tolerance failures in a specific radiation environment (high environmental levels because that seems to be a "conservative" approach), but one which is deemed unlikely to be encountered, the result appears to be a cost for the activity with little calculable gain in reliability.

If these are not clearly areas of uncertainty, then they at least seem to be activities in which the risks can be slanted, dependent on a particular evaluation of failure probabilities. Slanting is used in the manner of Fellner in [12] where the mathematical expectation of values is consistently adjusted to influence the quantity of product in one way or the other. Consider Figure 7. As indicated before in our calculation of an initial reliability goal using a constant value of product, P, the intersection of P with marginal cost (MC) gives a reliability level t

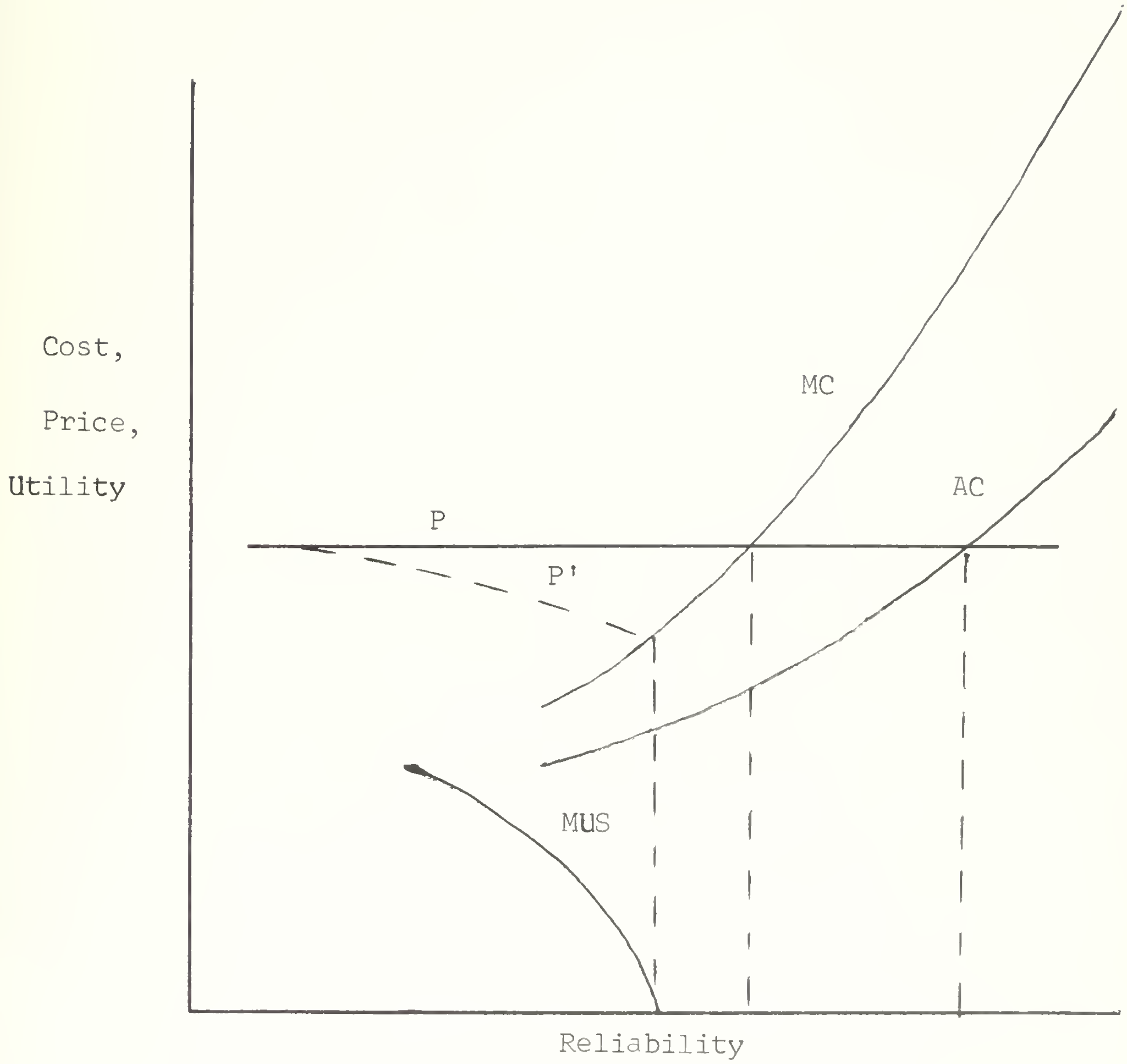

FIGURE 7. LEVELS OF OUTPUT

the left of the intersection with average cost (AC). Now if one maximized utility where the marginal utility of product was diminishing, the indicated level of reliability would be further to the left as illustrated by a marginal utility surplus (MUS) curve crossing the abscissa. This would be the same as saying that P decreased with reliability in the manner of curve P' to intersect the MC curve as shown. This is analogous to our previous contention that the assumption of constant price of product puts an upper bound on the reliability goal which one should set. Of course, it could happen that a decision maker assumed constant price and found the intersection with average cost, but slanted all his expectations in such a manner to select the reliability level indicated by correct use of MUS. But that would be happenstance, and he might slant to make matters worse.

Next consider activities (1d), (2b), and (4a):

(1d) Functional Design - Provisions for test in field

(2b) Packaging - Accessibility for test and repair/replace in field

(4a) Operational Procedures - Test and repair/replace cycle

These three activities are all closely interdependent and related to dormant failure rates or failures induced by the testing itself. In functional design, the circuits must be provided so that information can be gained by monitoring as to whether the equipment is good or bad. The surest way is to essentially cause the equipment to operate, but this would usually involve a severe design penalty to preclude significant degradation from the test itself. A visual test would not degrade the device but neither

would it give much information. Thus, the test procedure selected will not give perfect information, and there will likely be some degradation from the test itself. Also one must have a general idea as to how many monitoring operations will be conducted in the service life of the weapon in order to select compatible components. The manner of packaging the functional device within the weapon affects the same factors and together the two activities largely determine what the "a" factor for a monitoring operation will be. The operational procedures would seem more amenable to change, such as deciding on tests during the service life when the original plan had been to treat the weapon as a wooden bomb. But changing operational procedures in mid-life for a complex logistics system may entail very real and large costs.

The aspect of uncertainty seems most important in the plans for testing. In general, if one believes the prediction of constant, low dormant failure rate λ_1, checkout testing would not make sense until considerable time had elapsed since production because it would give little payoff. Essentially, the payoff is low because reliability for the lot can only be improved when bad devices are found and corrected. For Poisson failures, a test "good" does nothing to improve probability that the device will be good at some time later.[1] And few failures will have developed in a short time (hazard product, $\lambda_1 t_1$, is low). If there were significant

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[1] This is called the property of "memorylessness". For failures which occur in a truly random fashion, a test "good" does not change the probability of a failure occurring in the next time period.

degradation from testing, the result of a monitoring operation too soon (small t_1) would be degradation of reliability. The only valid reason for very frequent monitoring operations then would seem to be based on the likelihood of the stress factor in the intervening phases of the FTS becoming large for some reason.[1] If this did happen, it would probably be due to a poor development and qualification program for the equipment. Hopefully, stress factors significantly above unity should only pertain for the flight phase of an REB FTS.

Suppose one decides on a wooden bomb philosophy in an REB development program based on a fairly low predicted dormant failure rate and adopts a design which allows for no provisions for test of the AFD in the field not for access to the AFD for repair or replacement. Should information from the stockpile sampling or operational test programs indicate a wholly unexpected reliability degradation has occurred, the corrective action would then involve essentially replacement of the entire REB. Let us simplify this situation by saying that if things go as expected, the ex ante average reliability,

$\bar{R}$ = 96% will pertain.

---

[1]Recall that failure rate during any phase of index i

$$\lambda_i = k_i u_i$$

where k_i is the stress factor and u_i is the basic failure rate under conditions defined as standard. The k_i's during missile flight phases are very large compared to all other phases which are usually considered unity so that only the basic failure rate applies during non-flight phases. Events such as rough handling or testing may induce failures at a rate larger than the basic failure rate unless the equipment is rugged enough to withstand all these non-flight environments. The occurrence of such induced failures is equivalent to increase of the stress factor during that phase.

with some unexpected failure, and correction being very difficult to effect, the resultant ex post average might be 40%. Then consider that the utility for reliability is the shape shown in Figure 8. The shape is derived from the following considerations. From some level of significant damage capability marginal utility increases rapidly until above some higher level - before we get into our regime of reliability improvement effort - marginal utility will decrease. The shape of the curve at very low levels is not relevant here. If one assigned subjective probabilities to the likelihood of average reliability being the predicted 96% or the unpredicted almost complete failure of a component to occur, based on experience with programs of similar complexity, the judgment might be:

$$p = 5\% \text{ that } \overline{R} = 40\%$$
$$1 - p = 95\% \text{ that } \overline{R} = 96\%.$$

Then in a manner analogous to an insurance/betting utility analysis, treating reliability units as money, the internal average,

$$p \cdot (40\%) + (1 - p) \cdot (95\%) = 93.2\%,$$

represents the certainty equivalent of this program. But the utility shape is for one who prefers insurance, and the construction in Figure 8 shows one would be willing to pay an insurance premium, the difference between 93.2% and 76%, to hedge against the catastrophic 40% result. Converting all calculations into monetary terms, it seems likely under fairly wide ranges on these judgmental

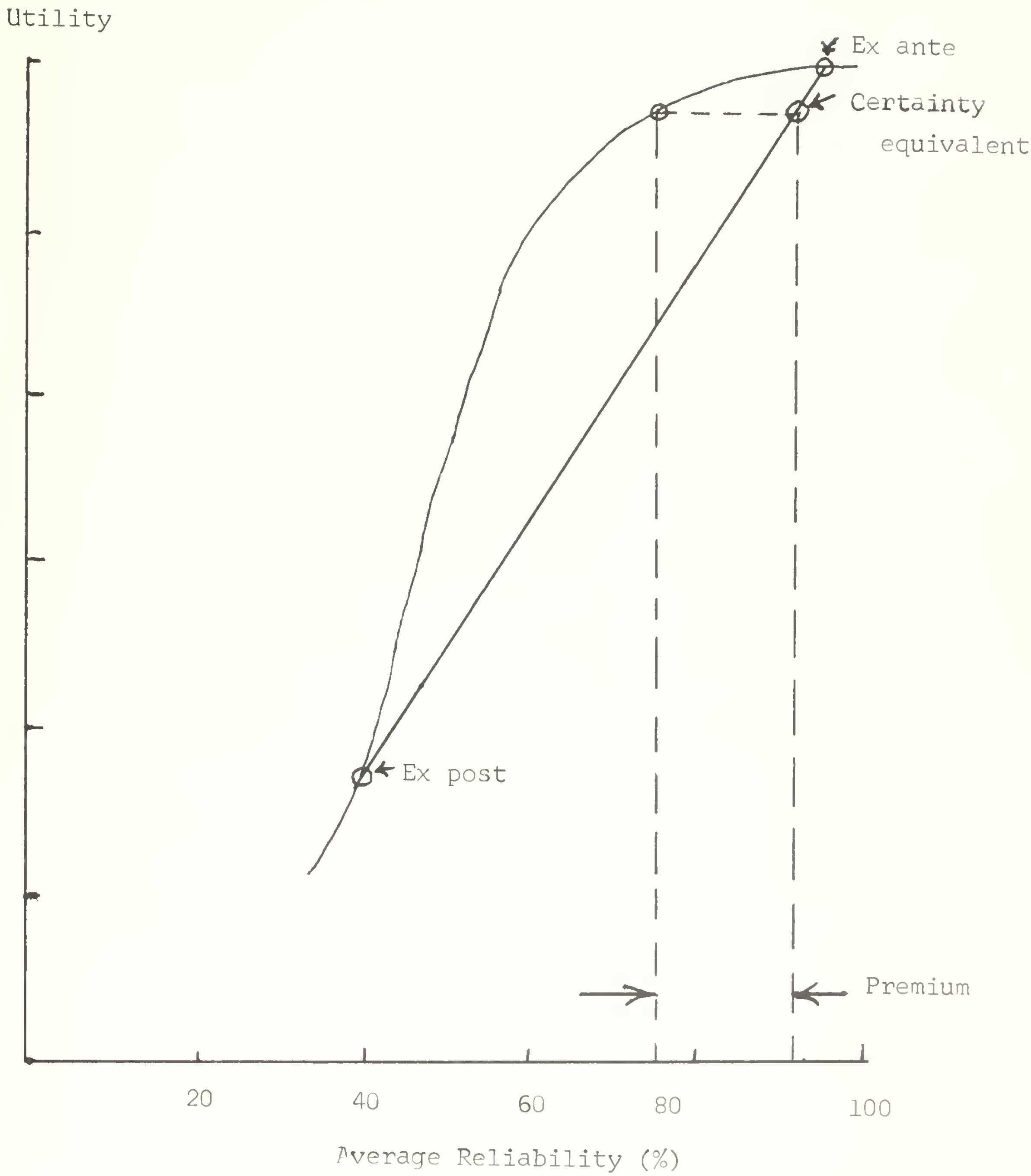

FIGURE 8. UTILITY OF RELIABILITY

values that one could make provisions for field test and repair at a cost which would not use up all the allowance for insurance premium.[1]

[1]Another way of hedging against the unpredicted failure mode is to use a redundant channel, utilizing different components and operating on a different physical principle. This reduces the likelihood of catastrophic reduction of reliability from unpredicted failures in a major way, but it does not eliminate the possibility.

V. ANSWERS TO QUESTIONS

If one had conducted an analysis along the lines indicated, could the questions raised in Section IA now be answered? It is apparent that there can be no answers which hold universally. Every weapon system must be studied on its own merits in much detail. Thus, we should avoid institutional arrangements that dictate test and maintenance philosophy before all the facts are known or lead to reliability specifications which are difficult to change for over-all benefit as the program progresses. Based on the REB example considered herein, we can make some general assertions concerning the questions if we stipulate enough conditions on each conclusion.

(1) There are certainly powerful inducements for decreeing that a wooden bomb be produced. Estimation of the present value of the cost of an involved test and maintenance activity over the service life of the weapon system will probably show up as a large sum whose avoidance looks very attractive. And the chance that any reliability payoff might be negative regardless of monetary cost if the wrong design alternatives or test procedures are selected is a discouraging outlook. The decision should depend on setting the marginal utility of average reliability equal to the marginal cost of obtaining it. If an agency finds the wooden bomb best on one program, it should not necessarily decree it as policy on all other programs. On the other hand, neither should an agency decree maintenance activities because

of arbitrary policy. This latter possibility is seen to be likely where program responsibility is split between semi-autonomous development and logistics commands. The logistics command tends to plan extensive maintenance activities for some reliability improvement without a clear picture of the over-all cost-benefit situation. The agency making decisions should have responsibilities which span the entire life of the weapon system.

(2) Chapter III herein covered the analysis concerning the nature and frequency of tests. The alternatives are many and complex, but external constraints of the system may in actuality limit the choices severely. Tests in general will be done when the REB is available at little cost. The nature of the tests should be dictated by the attempt to insure a correct answer to the question of good or bad. But this is a tradeoff with limiting degradation from the test procedure. From the monitoring operations there is the byproduct of some information leading to forecast of the approach of the wearout period. This should probably not be a significant factor in deciding the frequency of these system tests. This statement is made on the assumption that it takes a much more thorough intensive (probably destructive) examination to detect impending wearout conditions than would be efficient for a monitoring operation which had the purpose of detecting failures which have already occurred. The stockpile sampling program as a separate activity could more efficiently predict the approach of wearout.

(3) The values postulated in our analysis would say that one should hedge against unexpected failure modes in conjunction with the wooden bomb concept and pay the cost for a removable functional device. This would not necessarily be true for a lower cost weapon or one where the logistic cost of replacement of the entire weapon were smaller.

(4) This thesis indicates several considerations for lowering a reliability goal previously set without proper consideration of all economic factors. Neglect of economic analysis seems to suggest likelihood of the reliability goal being set too high. The many unknown factors and iterative calculations called for make proper decision seem very difficult, but in actuality there may be only a fairly small number of practical alternatives from which to choose. Again, a development agency with over-all program responsibility is suggested as a desirable institutional arrangement because of the necessity for authority to change reliability goals without undue administrative delay as new information becomes available.

(5) The last question, whether to develop a functional device based on a different physical principle, is really not different from consideration of other design alternatives except for the higher uncertainty attendant to a research program. For a given system effectiveness, the present value of the cost of the two programs is compared. However, the use of arbitrary rules rather than rational economic reasoning could dictate the wrong alternative. For example, if an agency had decreed the wooden

bomb concept for the entire service life in conjunction with a strict minimum reliability specification, an electronic device might be ruled out on technical feasibility grounds even though its choice might be a better alternative if a few maintenance cycles were allowed.

In summary, we have indicated in simple terms a rationale for the application of economic analysis to a very complex technical problem. The following quotation from Joan Robinson seems appropriate:

> The analysis can be extended to any degree of refinement, but the more complicated the question the more cumbersome the analysis. In order to know anything it is necessary to know everything, but in order to talk about anything it is necessary to neglect a great deal. [17]

APPENDIX 1

GENERALIZED LAGRANGE MULTIPLIER METHOD

A. Background

The method suggested in Chapter II for solving the problem of optimum allocation of reliability improvement activities is a simple application of the Generalized Lagrange Multiplier Method explained in reference [13]. The basic framework of the method will be outlined in this appendix in terms of the problem in this thesis.

The data for this problem would not normally be in the form of differentiable functions but a set of cost estimates corresponding to several discrete intensity levels of each activity. This use of Lagrange multipliers constitutes a technique whose goal is maximization of the total reliability improvement subject to a cost constraint. The domain of the function to be maximized is the set of reliability improvements possible, choosing one intensity level from each improvement activity.

B. Formulation

In our problem we have set up four different activities of index, $i = 1$ to 4. For each we can select an intensity level which would by itself result in a reliability improvement of $X_i = 1$, 2, or 3%. The set of possible strategies consists of all possible combinations of these activity levels. The over-all reliability improvement, X, from any strategy is called the payoff

function. Corresponding to the intensity level of each activity there is a cost, C_i, which pertains if that activity were the only one employed. The total cost, C, of any strategy is called the resource function. The problem to be solved is maximization of the payoff subject to a constraint on the resource.

As a subclass of the general problem, our problem is called a cell or separable problem in that the activities are considered to be independent of each other (over the small range of variables in question) such that the over-all payoff is the sum of the individual payoffs, and the over-all cost is the sum of the individual costs. The problem to be solved in this case is to find a strategy, one intensity level from each activity, which maximizes the reliability improvement subject to a constraint on cost:

$$\text{Maximize:} \quad X = \sum X_i$$

$$\text{Subject to:} \quad \sum C_i = C.$$

C. Main Theorem

The main theorem of reference [13] in our terminology and for only one constraint is as follows: Given that

(1) All Lagrange multipliers, μ, are nonnegative real numbers

(2) Strategy S^* maximizes the unconstrained function $X - \mu C$;

then it follows that S^* maximizes X over all strategies such that $C \leq C(S^*)$.

This theorem says that for any choice of nonnegative μ, if a maximum of the unconstrained Lagrangean function

$$X - \mu C$$

can be found, then the solution is a solution to the constrained maximization problem where the constraint is the cost expended in achieving the unconstrained solution. According to the theorem, one can simply choose an arbitrary set of nonnegative μ's, find an unconstrained maximum of the modified payoff function, $X - \mu C$, and one has as a result a solution to a constrained problem. Different choices of μ's lead to different cost levels. The method does not guarantee that an answer can be found, but asserts that if an answer can be found it will indeed be optimum.

In applying the main theorem to the cell problem it is shown that maximizing

$$X_i - \mu C_i$$

for each activity independently of strategy choices in other activities, and summing the payoffs and costs for each activity (for the strategy that maximized the Lagrangean for that activity); results in maximizing $\sum X_i$ with cost constraint equal to $\sum C_i$, both summations over the strategy produced by the procedure.

The theorem says nothing about the manner in which one obtains the maxima of the unconstrained Lagrangean functions. In our case where we do not have analytic functions it cannot be done by finding zeroes of derivatives. It is done in this simple example by listing all discrete possibilities and picking the maximum result.

APPENDIX 2

ACTIVITIES FOR IMPROVEMENT OF INITIAL RELIABILITY

From the categorization in Section ID of all processes for reliability improvement, four activities are chosen as particularly applicable to initial (when new) reliability. The four are discussed below as applied to the case of an electronic arming and fuzing device (AFD) in the re-entry body (REB) of a strategic missile.

(1) Reduction of AFD Complexity

The AFD was made complex to provide a level of accuracy in sensing height of burst (HOB) above target. The very nature of electronic device failure rate versus complexity then dictates a lower reliability than a less complex device. The HOB accuracy contributes to determination of a radius of effect (R_E) for warhead explosive yield to kill a target of given hardness. Accuracy in terms of the circular error probable (CEP) of the burst is also affected because of the interaction between HOB dispersion and the re-entry trajectory. R_E and CEP together determine the probability of kill (P_K) for a single target $\left(P_K \cong 1 - \frac{1}{2}^{\left[R_E/CEP\right]^2}\right)$. Thus the functional relations exist for calculating opportunity costs due to decreased P_K from lowering complexity if we assign a worth to the REB and also decide on how this value is divvied up among the units of P_K of the basic design.

Sophisticated methods looking at effectiveness over a complete target complex may be dictated. In lowering complexity there will be an offsetting saving (to the opportunity costs) in pecuniary costs of development and production, but they may not be too significant in comparison. (It should be noted that increased complexity can result in higher reliability if the complexity is required to overcome an enemy countermeasure environment. We are assuming that increased complexity is only for the purpose of improving performance - better accuracy, given no failure.)

(2) Redundancy

The design options under this activity involve putting entire functional channels in parallel with the basic series functional circuit or selected components or groups of components in parallel with others. Components with suspected high fallibility would of course be selected for redundancy. The entire channel redundancy is normally a choice between single or dual because the marginal return in reliability rapidly becomes very small and may even go negative due to cross-channel failure paths. We are assuming here the channel or component is duplicated with another identical one. Consideration of using a different type functional circuit as a redundant one involves hedging against uncertainty in predicting reliability and will be discussed in Chapter IV. The major costs of this activity are likely to be the opportunity costs represented by the weight and space allocation robbed from the warhead which would result in lower explosive yield. Since R_E is roughly proportional to $W^{1/3}$ where W = yield of warhead, the opportunity cost can be calculated as done for activity #1. The

pecuniary costs in this case would be additive rather than off-setting but probably not too significant. (Another possible cost is in the area of nuclear safety, the provision of low probability of an unwanted detonation, and should be mentioned. It is sometimes assumed that increased reliability means reduced safety because a series of functional safety devices would indeed imply reduced reliability compared to their simple removal. However, much can be done in the way of devices which serve both operating and safety functions, and it is by no means clear that a proliferation of series elements will insure maximum safety. In any event an almost absolute safety criterion is enforced for any design option. Any differences in probability of unwanted detonation between different designs are essentially not calculable - the differences between very small numbers. The cost of an unwanted nuclear detonation is unknown except to say it is unacceptably high. Significant costs for providing safety are difficult to separate from the other costs of development. For these reasons an estimate of costs involving safety will not be included.)

(3) Provision for Integrated Test in Manufacture

From the standpoint of efficiency in packaging the functional parts in the REB - where efficiency means allowing more weight and space allocation for the warhead and more freedom of design for the REB structure - it is generally better to split up the AFD and tuck it here and there to best utilize the available locations. But this splitting up means addition of cable runs with inherent

fault potentials and a very much more difficult job of testing the AFD during development, evaluation, and qualification programs in its operational configuration. Thus to gain reliability in a design which is not single unit, we can choose designs which come closer to being single unit. As in activity #2 the robbing of space and weight allocation will result in lower yield. The effect might even be severe enough to decrease accuracy due to limitations on REB structure design options which would influence aerodynamic characteristics during re-entry. In any event the opportunity costs due to reduced P_K can be calculated. One would expect some offsetting savings in the test programs. Experience on earlier designs should enable engineers to make reasonable estimates of the ΔR to be gained from different degrees of movement toward a single unit design.

(4) Manufacturing and Quality Control

This activity would apply to a design resulting from any combination of intensity levels in the first three activities. It would consist of actions such as:

(a) Use of high reliability parts, attained by careful design, intensive controls during manufacture, extensive testing for statistical confidence in the design, and screening manufactured parts by complete testing of parameters.

(b) Intensive process controls during manufacture and quality control tests of the end item.

The costs of this activity are the pecuniary costs of different intensity levels. Actually the cost for a given ΔR would be dependent on what combination of the first three activities was embodied in the design and program changes proposed. On the other hand, over a limited range of design and program changes, considering the C_i versus X_i relationship for this activity as independent of the other activities might not be a bad approximation, bearing in mind we are looking at limited changes to a basic program which already includes a considerable amount of this kind of activity.

LITERATURE CITED

1. Myers, Richard H., Kam L. Wong, Harold M. Gordy, editors, Reliability Engineering for Electronic Systems. (New York: Wiley, 1964).

2. ARINC Research Corporation, William von Alven, editor, Reliability Engineering. (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1964).

3. "Reliability Stress and Failure Rate Data For Electronic Equipment", MIL-HDBK-217A. (Washington, D.C.: Department of Defense, 1965).

4. Martin Matietta Corporation, D. F. Cotrell, T. R. Gagnier, E. W. Kimball, et al., "Dormant Operation and Storage Effects on Electronic Equipment and Part Reliability", Rome Air Development Center, Technical Report No. RADC-TR-67-307. Griffiss Air Force Base, New York: 1967.

5. Brender, David M., "The Uncertainty of System Failure-Rate Predictions", IEEE Transactions on Reliability, Vol. R-16, No. 2, pp. 75-81, 1967.

6. Bowman, Edward H. and Robert B. Fetter, Analysis for Production Management. (Homewood, Illinois: Richard D. Irwin, Inc., 1961).

7. Viner, Jacob, "Cost Curves and Supply Curves", in American Economic Association, Readings in Price Theory. (Homewood, Illinois: Richard D. Irwin, Inc., 1952), pp. 198-232. Reprinted from Zeitschrift für National ökonomie, Vol. III, pp. 23-46, 1932.

8. Cohen, Kalman J., and Richard M. Cyert, Theory of the Firm: Resource Allocation in a Market Economy. (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1965).

9. Liebhafsky, H. H., The Nature of Price Theory. (Homewood, Illinois: The Dorsey Press, Inc., 1963).

10. Knight, Frank H., Risk, Uncertainty and Profit. (New York: Sentry Press, Reprint, 1964).

11. Friedman, Milton and L. J. Savage, "The Utility Analysis of Choices Involving Risk" in American Economic Association, Readings in Price Theory. (Homewood, Illinois: Richard D. Irwin, Inc., 1952). pp. 57-96. Reprinted from the Journal of Political Economy, Vol. LVI, pp. 279-304, 1948.

12. Fellner, William, Probability and Profit - A Study of Economic Behavior Along Bayesian Lines. (Homewood, Illinois: Richard D. Irwin, Inc., 1965).

13. Everett, Hugh, III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources", Operations Research, Vol. 11, pp. 399-417., 1963.

14. Ferguson, Robert L., "No-Test, No-Maintenance Economics", IEEE Transactions on Aerospace Support Conference Procedures, Vol. AS-1, pp. 889-894, 1963.

15. Samuelson, Paul A., Foundations in Economic Analysis.(New York: Atheneum, 1967).

16. Weiss, George H., "Optimal Periodic Inspection Programs for Randomly Failing Equipment", Journal of Research of the National Bureau of Standards - B. Mathematics and Mathematical Physics, Vol. 67B, No. 4, pp. 223-228, 1963.

17. Robinson, Joan, "Rising Supply Price", in American Economic Association, Readings in Price Theory. (Homewood, Illinois: Richard D. Irwin, Inc., 1952). p. 241. Reprinted from Economica, New Series, Vol. VIII, p. 8, 1941.